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Symmetry and symmetry breaking for ground state solutions of some strongly coupled elliptic systems

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Abstract

We consider the ground state solutions of the Lane–Emden system with Hénon-type weights $-\Delta u = |x|^\beta |v|^{q-1}v$, $-\Delta v = |x|^\alpha |u|^{p-1}u$ in the unit ball B of $\mathbb{R}^N$ with Dirichlet boundary conditions, where $N \geqslant 1$, $\alpha, \beta \geqslant 0$, $p, q > 0$ and $1/(p+1) + 1/(q+1) > (N-2)/N$. We show that such ground state solutions u, v always have definite sign in B and exhibit a foliated Schwarz symmetry with respect to a unit vector of $\mathbb{R}^N$. We also give precise conditions on the parameters α, β, p and q under which the ground state solutions are not radially symmetric.

Keywords: Hamiltonian elliptic systems; Hénon weights; Supercritical problems; Schwarz foliated symmetry; Symmetry breaking; Ground state solution

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Contents

1. Introduction

In this paper we are interested on the qualitative properties of the ground state solutions of the strongly coupled system

$$\begin{cases} -\Delta u = |x|^\beta |v|^{q-1} v, \quad -\Delta v = |x|^\alpha |u|^{p-1} u \quad \text{in } B, \\ u, v = 0 \quad \text{on } \partial B. \end{cases} \tag{1.1}$$

Here B stands for the open unit ball in $\mathbb{R}^N$, $N \geqslant 1$; $\alpha, \beta \geqslant 0$; $p, q > 0$. We refer to this system as a Lane–Emden system with Hénon-type weights.

The first mathematical work on the Hénon equation [20]

$$-\Delta u = |x|^\alpha |u|^{p-1} u, \quad x \in B, \text{ with } u = 0 \text{ on } \partial B, \ \alpha > 0, \ p > 1 \tag{1.2}$$

was published by Ni [24] who observed that the presence of the weight modifies the consequences of the Pohožaev identity. Basically, a new critical exponent arises for the non-existence of classical solutions. This exponent is also the exact threshold for the existence of radial solutions.

Ni's results were later extended to the corresponding equation involving the biharmonic operator under Dirichlet boundary condition by Dalmasso [12]. Precisely, Dalmasso extended the existence and non-existence results to the fourth-order equation with Dirichlet boundary condition,

$$\Delta^2 u = |x|^\alpha |u|^{p-1} u \quad \text{in } B, \qquad u = 0, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B. \tag{1.3}$$

Recently, de Figueiredo et al. [15] presented some contributions to (1.2) and also to (1.3) under both Navier or Dirichlet boundary conditions.

After Ni's pioneering work, a lot of effort has been devoted to the study of (1.2). Some remarkable results that we would like to mention are those in [7,9,25,30,29].

Since $|x|^\alpha$ increases with $|x|$, neither reflection nor symmetrization arguments can be applied to (1.2) to prove radial symmetry of either positive or ground state solutions. In fact, it was proved in [30] that the radial symmetry holds for small values of α while it breaks down when α is sufficiently large. However, as shown in [25,29], the ground state solutions still possess a residual symmetry, the so-called foliated Schwarz symmetry (see below).

A first attempt to study this phenomenon for (1.1) was done by Calanchi and Ruf [8]. Further contributions on this type of problem, in the case of Hamiltonian systems, were presented by Liu and Yang [18], see also de Figueiredo et al. [14].

Concerning the non-existence of solutions, we recall here a result in [8, Theorem 2.a)] (see also [14,18]) according to which (1.1) has no positive solutions ($u, v > 0$ in B) in case $\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} \leqslant N-2$, $p>0$, $q>0$, $N \geqslant 3$; this turns out to be a consequence of a suitable Pohožaev type identity. We prove that the hyperbola

$$\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} = N-2 \tag{1.4}$$

is the exact threshold for the existence of positive solutions of (1.1). Namely, we prove that (1.1) has a positive classical solution under the following general condition:

(H1) $\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} > N-2$.

Theorem 1.1. *Assume* (H1) *and* $pq \neq 1$. *Then* (1.1) *has a radial positive* ($u, v > 0$ *in* B), *classical solution* $(u, v) \in C^{2,\beta^*}(\overline{B}) \times C^{2,\alpha^*}(\overline{B})$ *with*

$$\alpha^* = \begin{cases} \min\{1, \alpha, p\}, & \text{if } \alpha < 1 \text{ or } p < 1; \\ \text{any } 0 < \gamma < 1, & \text{if } \alpha \geqslant 1 \text{ and } p \geqslant 1; \end{cases}$$
$$\beta^* = \begin{cases} \min\{1, \beta, q\}, & \text{if } \beta < 1 \text{ or } q < 1; \\ \text{any } 0 < \gamma < 1, & \text{if } \beta \geqslant 1 \text{ and } q \geqslant 1. \end{cases}$$

Furthermore, u *and* v *are both strictly radially decreasing.*

The proof will be presented in Section 2. In contrast with [8, Theorem 2.b)], here we do not impose any additional restrictions or compatibility relations between α, β and the powers p and q.

We recall that in the sublinear case $pq < 1$, as proved in [23, Theorem 4.1], a uniqueness result holds for classical positive solutions of (1.1).

Next we turn our attention to the existence and properties of *ground state*, also called *least energy*, solutions of (1.1). The precise meaning of ground state solutions will be given in Section 3. We anticipate that such definition will require that the couple (p, q) lies below the so-called critical hyperbola associated to the (non-weighted) system, namely

(H2) $\frac{N}{p+1} + \frac{N}{q+1} > N-2$.

In Section 3, we will establish the following result. For brevity, we omit the conclusions on the regularity of the solutions, which are the same as in Theorem 1.1 above (cf. Lemma 3.6).

Theorem 1.2. *Assume* (H2) *and* $pq \neq 1$. *Then* (1.1) *has a ground state* (*classical*) *solution. In addition, any ground state solution of* (1.1) *has definite sign, i.e. either* $u, v > 0$ *in* B *or* $u, v < 0$ *in* B.

In the case where $\alpha = \beta = 0$, the radial symmetry of the ground state solutions was established in [5, Theorem 1.6]. As mentioned above, such a result is no longer expected for large values of the parameters α or β in (1.1). On the other hand, we recall that a continuous function $w : B \to \mathbb{R}$ is called *foliated Schwarz symmetric* with respect to a unit vector $e \in \mathbb{R}^N$ if w is axially symmetric with respect to the axis $\mathbb{R}e \subset \mathbb{R}^N$ and is nonincreasing in the polar angle $\theta = \arccos(\frac{x}{|x|} \cdot e)$. In Section 3 we will prove the following.

Theorem 1.3. *Assume* (H2) *and* $pq \neq 1$. *For any* $\alpha, \beta \geqslant 0$, *every* (*positive*) *ground state solution* (u, v) *of* (1.1) *is such that* u *and* v *are both foliated Schwarz symmetric with respect to the same unit vector* e *in* $\mathbb{R}^N$. *Moreover, either* u *and* v *are radially symmetric or* u *and* v *are strictly decreasing in* $\theta \in [0, \pi]$ *for* $0 < |x| < 1$.

A deeper analysis of the problem (1.1) consists in establishing under which conditions symmetry breaking occurs. We observe that in the sublinear case $pq < 1$ one cannot expect such a phenomenon, since positive solutions are unique and therefore radially symmetric. Without loss of generality, we assume that $\beta \leqslant \alpha$. For definiteness, we let $\alpha \to \infty$ and will say that *symmetry breaking occurs* when there exists $\alpha_0 > 0$ such that for $\alpha \geqslant \alpha_0$, no ground state solution of (1.1) is radially symmetric.

We consider first the case where α and β are comparable to each other. In this case, we obtain a full extension of the main result in [30].

Theorem 1.4. (α and β comparable) *Assume* (H2), $pq > 1$ *and* $N > 1$ *and that there exists* $C > 0$ *such that* $\beta \leqslant \alpha \leqslant C\beta$ *as* $\alpha \to \infty$. *Then symmetry breaking occurs.*

We mention that the conclusion of Theorem 1.4 is likely to hold also in the one-dimensional case $N = 1$, by using some recent ideas in [7]. On the other hand, the case where α and β are no longer comparable, that is $\beta = \mathrm{o}(\alpha)$ as $\alpha \to \infty$, seems to be more delicate to handle. Indeed, as shown in [30], the simplest way to prove symmetry breaking for (1.2) is to observe that the ground critical level of (1.2) is asymptotically strictly smaller than the action on any radial solution; for the system (1.1) the situation is more tricky since both the corresponding ground critical levels, the radial one and the global one, may grow asymptotically at the same rate, cf. Proposition 2.9 and also Remark 3.16. Incidentally, our estimate in Proposition 2.9 corrects the estimate [8, Theorem 2c].

As we will show, our proof of Theorem 1.4 allows us to deal with a number of situations depending on the ratio β/α, at the price of a more restrictive assumption than (H2) (see e.g. case (i) in Theorem 1.6 below). In another direction, following an idea introduced in [28] and developed in [4,27], we apply a Lyapunov–Schmidt type reduction which enables us to extend to systems a strategy used in [30] for the single equation case and which is based on the computation of the second derivative of the underlying energy functional.

Theorem 1.5. ($\beta = \mathrm{o}(\alpha)$ and $\beta \to \infty$) *Assume* (H2), $pq > 1$, $p \geqslant 1$, $q \geqslant 1$, $N \geqslant 3$ *and that* $\beta = \mathrm{o}(\alpha)$ *and* $\beta \to \infty$ *as* $\alpha \to \infty$. *Then symmetry breaking occurs.*

We emphasize that, for instance, the previous statement means that for any positive sequences α_n, β_n with $\beta_n \to \infty$ and $\beta_n/\alpha_n \to 0$ (so that also $\alpha_n \to \infty$), no ground state solution of (1.1) with $\alpha = \alpha_n$ and $\beta = \beta_n$ is radially symmetric for n sufficiently large.

We also state a result which particularizes the previous theorem in the case where one of the exponents in (1.1) is fixed and the other one moves to infinity. Concerning assertion (i) below, it is worthwhile to observe that (H2) can also be written in the equivalent form $0 < 2(p+1)(q+1) - N(pq-1)$.

Theorem 1.6. (β fixed) *Assume* (H2), $pq > 1$, $N \geqslant 1$ *and* $\beta \geqslant 0$. *Then symmetry breaking occurs in any of the following two cases*:

(i) (p, q) *is sufficiently close to the critical hyperbola, namely*

$$2(p+1)(q+1) - N(pq-1) < (p+2)(q+1);$$

or

(ii) β *is taken sufficiently large and* $p \geqslant 1$, $q \geqslant 1$, $N \geqslant 3$.

We point out that, under the general subcriticality assumption (H2), the case where β is fixed but small (say $\beta = 0$) and $\alpha \to \infty$ remains an interesting open problem. We refer to Remark 3.16 for further comments.

The remaining of the paper is devoted to the proof of the theorems mentioned so far. The presentation in Sections 2.1 and 3.1 is close to the one in [5]. The key point is played here by the regularity result in Lemma 2.8, which is in turn based on Proposition 2.1; we can formulate the problem in such a way that existence, regularity, positivity and foliated Schwarz symmetry of the solutions are deduced smoothly.

The analysis of the symmetry breaking requires more technical effort and is presented in full detail along Sections 2.2 and 3.3. Since we use several different approaches to the problem, we found it useful to include in Section 4 an appendix which summarizes them, so as to ease the reading of the paper and compare it with the existing literature.

2. Radial solutions

2.1. Existence and regularity

We obtain our existence result on positive radial solutions by searching for least energy solutions amongst all radial solutions. To that purpose, let us introduce some Sobolev spaces.

For each $\gamma \geqslant 0$ and $s > 1$, we set

$$E_{s,\gamma,\mathrm{rad}} = \left\{ u \in W^{2,s}_{\mathrm{rad}}(B) \cap W^{1,s}_0(B) \colon \int_B |\Delta u|^s |x|^{-\gamma}\, dx < +\infty \right\}.$$

Endowed with the norm $\|u\|_{s,\gamma}$ defined by

$$\|u\|_{s,\gamma} = \left(\int_B |\Delta u|^s |x|^{-\gamma}\, dx \right)^{1/s}, \tag{2.1}$$

$E_{s,\gamma,\mathrm{rad}}$ is a Banach space. We first prove useful embeddings.

Proposition 2.1. *The following assertions hold.*

(1) *Every function* $u \in E_{s,\gamma,\mathrm{rad}}$ *is almost everywhere equal to a function* $U \in C^1(\overline{B} \backslash \{0\})$. *In addition, for any multi-index* α *with* $|\alpha| = 2$, $D^\alpha U(x)$ *exists, almost everywhere, in the classical sense.*

(2) *If* $N-2s-\gamma>0$*, there exists* $C>0$ *such that for all* $u\in E_{s,\gamma,\mathrm{rad}}$ *and every* $x\in B\backslash\{0\}$*,*

$$\left|U(x)\right|\leqslant C\|u\|_{s,\gamma}|x|^{-(\frac{N-2s-\gamma}{p})},\tag{2.2}$$

where U *is given by* (1). *In particular, if* $N-2\frac{q+1}{q}-\frac{\beta}{q}>0$*, then* $E_{\frac{q+1}{q},\frac{\beta}{q},\mathrm{rad}}$ *is compactly imbedded in* $L^{p+1}(B,|x|^\alpha)$ *for every* $p,q>0$ *satisfying* (H1).

(3) *If* $N-2s-\gamma=0$*, then there exists* $C>0$ *such that for all* $u\in E_{s,\gamma,\mathrm{rad}}$ *and every* $x\in B\backslash\{0\}$*,*

$$\left|U(x)\right|\leqslant C\|u\|_{s,\gamma}\left|\log|x|\right|^{\frac{s-1}{s}},\tag{2.3}$$

where U *is given by* (1). *In particular, if* $N-2\frac{q+1}{q}-\frac{\beta}{q}=0$*, then* $E_{\frac{q+1}{q},\frac{\beta}{q},\mathrm{rad}}$ *is compactly imbedded in* $L^{p+1}(B,|x|^\alpha)$ *for all* $p>0$ *and all* $\alpha\geqslant 0$.

(4) *If* $N-2s-\gamma<0$*, then* $E_{s,\gamma,\mathrm{rad}}$ *is continuously imbedded in* $C(\overline{B})$. *In particular, if* $N-2\frac{q+1}{q}-\frac{\beta}{q}<0$*, then* $E_{\frac{q+1}{q},\frac{\beta}{q},\mathrm{rad}}$ *is compactly imbedded in* $L^{p+1}(B,|x|^\alpha)$ *for all* $p>0$ *and all* $\alpha\geqslant 0$.

Proof. Statement (1) follows from [15, Theorem 1.1 (1)], since $E_{s,\gamma,\mathrm{rad}}\subset W^{2,s}_{\mathrm{rad}}(B)$. As for the compactness of the imbedding, we observe that (H1) is equivalent to the inequality

$$\alpha+(p+1)\left(\frac{-N+2\frac{q+1}{q}+\frac{\beta}{q}}{\frac{q+1}{q}}\right)>-N.$$

The assertions (2), (3) and (4) can be proved similarly arguing as in the proof of [8, Proposition 9]. We stress that the vanishing $w(0)=0$ at [8, p. 123, l. 1], with $N\geqslant 2$, is guaranteed by the fact that $E_{s,\gamma,\mathrm{rad}}\subset W^{2,s}_{\mathrm{rad}}(B)$, $s>1$, thanks also to [15, Theorem 2.2 and Lemma 2.5]. When $N=1$, which implies $N-2s-\gamma<0$, we do not use w as in [8, p. 123, l. 1]; instead, we have, for $t\in[0,1]$, the identity

$$u(t)=\int_1^t u'(\theta)\,d\theta=\int_1^t\int_0^\theta u''(\sigma)\,d\sigma-u'(0)(1-t),$$

and we can estimate $|u'(0)|$ by $\|u\|_{s,\gamma}$ as in [19]. We also refer to [15, Eq. (2.12)]. □

We next rewrite (1.1) as

$$\Delta\left(|x|^{-\frac{\beta}{q}}|\Delta u|^{\frac{1}{q}-1}\Delta u\right)=|x|^\alpha|u|^{p-1}u\quad\text{in } B,\ \text{with } u,\Delta u=0\text{ on }\partial B.\tag{2.4}$$

In the sequel, for short, E_{rad} stands for $E_{\frac{q+1}{q},\frac{\beta}{q},\mathrm{rad}}$. It follows from Proposition 2.1 that the embedding $E_{\mathrm{rad}}\hookrightarrow L^{p+1}(B,|x|^\alpha)$ is compact whenever (H1) holds. This motivates the following definition.

Definition 2.2. Assume (H1). We say that u is a weak radial solution of (2.4) if u is a critical point of the functional $J_{\mathrm{rad}}:E_{\mathrm{rad}}\to\mathbb{R}$ defined by

$$J_{\mathrm{rad}}(u) = \frac{q}{q+1} \int_B |\Delta u|^{\frac{q+1}{q}} |x|^{-\frac{\beta}{q}}\, dx - \frac{1}{p+1} \int_B |u|^{p+1} |x|^{\alpha}\, dx,$$

that is, if $u \in E_{\mathrm{rad}}$ satisfies

$$\int_B |\Delta u|^{\frac{1}{q}-1} \Delta u \Delta\varphi |x|^{-\frac{\beta}{q}}\, dx = \int_B |u|^{p-1} u\varphi |x|^{\alpha}\, dx, \quad \forall \varphi \in E_{\mathrm{rad}}.$$

We will see in Lemma 2.8 below that a weak radial solution u of (2.4), in the sense of Definition 2.2, produces a classical solution (u, v) of (1.1) with $-v = |x|^{-\frac{\beta}{q}} |\Delta u|^{\frac{1}{q}-1} \Delta u$.

Remark 2.3. One might compare the Sobolev space E_{rad} defined as above with the one introduced in [8, p. 113]. It turns out that these spaces coincide if and only if there exists a positive constant C such that

$$\int_B \left|D^2 u\right|^{\frac{q+1}{q}} |x|^{-\frac{\beta}{q}}\, dx \leqslant C \int_B |\Delta u|^{\frac{q+1}{q}} |x|^{-\frac{\beta}{q}}\, dx, \quad \forall u \in E_{\mathrm{rad}}. \tag{2.5}$$

However, it is not clear whether this holds for general $\beta \geqslant 0$ (it clearly holds for $\beta = 0$). We stress that, in contrast with [8], here we do not need at all that $qN > \beta$.

Remark 2.4. In case $pq = 1$, define

$$\lambda_{1,\alpha,\beta,q} := \inf_{u \in E_{\mathrm{rad}},\, u \neq 0} \frac{\int_B |\Delta u|^{\frac{q+1}{q}} |x|^{-\frac{\beta}{q}}\, dx}{\int_B |u|^{\frac{q+1}{q}} |x|^{\alpha}\, dx}. \tag{2.6}$$

If $\lambda_{1,\alpha,\beta,q} > 1$ then (2.4) has no nontrivial weak radial solutions. On the other hand, if $\lambda_{1,\alpha,\beta,q} \leqslant 1$ then $J_{\mathrm{rad}}(u) = 0$ for any weak radial solution $u \in E_{\mathrm{rad}}$; in particular the value $J_{\mathrm{rad}}(u)$ does not distinguish weak solutions of (2.4).

In virtue of the previous remark, and since we will be dealing with least energy radial solutions of (2.4), in the sequel we always assume that $pq \neq 1$.

Definition 2.5. Assume (H1) and $pq \neq 1$. We say that $u \in E_{\mathrm{rad}} \backslash \{0\}$ is a *least energy radial solution* for (1.1) if J_{rad} attains its smallest nonzero critical value at u.

In the sequel we assume that (H1) and $pq \neq 1$ hold. We denote by $N_{J_{\mathrm{rad}}}$ the Nehari manifold associated to the functional J_{rad}, namely

$$N_{J_{\mathrm{rad}}} := \left\{ u \in E_{\mathrm{rad}} \backslash \{0\}\colon\ J'_{\mathrm{rad}}(u)u = 0 \right\},$$

and we introduce the minimization problems

$$c_{J_{\mathrm{rad}}} := \inf_{u \in N_{J_{\mathrm{rad}}}} J_{\mathrm{rad}}(u) \tag{2.7}$$

and

$$m_{p,q,\alpha,\beta,\mathrm{rad}} := \inf\left\{\int_B |\Delta u|^{\frac{q+1}{q}} |x|^{-\frac{\beta}{q}}\,dx\colon\ u \in E_{\mathrm{rad}},\ \int_B |u|^{p+1}|x|^{\alpha}\,dx = 1\right\}. \tag{2.8}$$

We observe that if $m_{p,q,\alpha,\beta,\mathrm{rad}}$ is achieved then $1/(m_{p,q,\alpha,\beta,\mathrm{rad}})^{q/(q+1)}$ is the optimal constant for the imbedding of E_{rad} in $L^{p+1}(B,|x|^{\alpha})$. The norm in the latter space will be denoted by $|\ |_{p+1,\alpha}$.

Once our framework is settled as above, one can deduce Theorem 1.1 by arguing exactly as in [5, Section 2]. For completeness, we indicate here the key steps of the argument.

Lemma 2.6. *The following assertions hold.*

(1) *The minimization problems* (2.7) *and* (2.8) *are equivalent in the sense that*:
  - (i) *Given a minimizing sequence* $(u_n) \subset N_{J_{\mathrm{rad}}}$ *for* (2.7), $(|u_n|_{p+1,\alpha}^{-1} u_n)$ *is a minimizing sequence for* (2.8).
  - (ii) *Given a minimizing sequence* $(\bar u_n)$ *for* (2.8), $(\|\bar u_n\|^{\frac{q+1}{pq-1}} \bar u_n) \subset N_{J_{\mathrm{rad}}}$ *is a minimizing sequence for* (2.7).
  - (iii) *We have the equality*

$$c_{J_{\mathrm{rad}}} = \frac{pq-1}{(p+1)(q+1)} m_{p,q,\alpha,\beta,\mathrm{rad}}^{\frac{q(p+1)}{pq-1}}. \tag{2.9}$$

  - (iv) *The optimal constant* $m_{p,q,\alpha,\beta,\mathrm{rad}}$ *is attained if and only if* $c_{J_{\mathrm{rad}}}$ *is attained. In addition, if* $\bar u$ *is a solution for* (2.8), *then* $\|\bar u\|^{\frac{q+1}{pq-1}} \bar u = m_{p,q,\alpha,\beta,\mathrm{rad}}^{\frac{q}{pq-1}} \bar u$ *is a solution for* (2.7). *Conversely, if* u *is a solution for* (2.7), *then* $|u|_{p+1,\alpha}^{-1} u$ *is a solution for* (2.8).

(2) *The optimal constant* $m_{p,q,\alpha,\beta,\mathrm{rad}}$ *is attained, i.e., there exists* $u \in E_{\mathrm{rad}}$ *such that* $|u|_{p+1,\alpha} = 1$ *and* $\|u\|^{\frac{q+1}{q}} = m_{p,q,\alpha,\beta,\mathrm{rad}}$.

(3) *If* $u \in N_{J_{\mathrm{rad}}}$ *is such that* $J_{\mathrm{rad}}(u) = c_{J_{\mathrm{rad}}}$, *then* u *is a least energy radial solution for* (1.1). *Conversely, if* u *is a least energy radial solution for* (1.1) *then* $J_{\mathrm{rad}}(u) = c_{J_{\mathrm{rad}}}$.

Proof. It is enough to follow the proof of [5, Lemmas 2.3 and 2.5]. Here we take into account that E_{rad} is compactly imbedded in $L^{p+1}(B,|x|^{\alpha})$, which is guaranteed by Proposition 2.1. □

Lemma 2.7. *Let* $u \in N_{J_{\mathrm{rad}}}$ *be such that* $J_{\mathrm{rad}}(u) = c_{J_{\mathrm{rad}}}$. *Then* $u, -\Delta u > 0$ *in* B, *or else* $u, -\Delta u < 0$ *in* B.

Proof. One can argue as in the proof of [5, Lemma 2.6]. The crucial point here is that given $u \in E_{\mathrm{rad}}$, then the strong solution w of the linear problem

$$-\Delta w = |\Delta u| \quad \text{on } B,\ \text{with } w = 0 \text{ on } \partial B$$

is also an element in E_{rad}. This feature displays the convenience of working with the space E_{rad} as defined above. □

Lemma 2.8. *Assume* (H1). *Let $u \in E_{\mathrm{rad}}$ be a weak radial solution of* (2.4) *in the sense of Definition* 2.2 *and set $-v = |x|^{-\frac{\beta}{q}}|\Delta u|^{\frac{1}{q}-1}\Delta u$. Then $(u, v) \in C^{2,\beta^*}(\overline{B}) \times C^{2,\alpha^*}(\overline{B})$ is a classical solution of* (1.1) *with*

$$\alpha^* = \begin{cases} \min\{1, \alpha, p\}, & \text{if } \alpha < 1 \text{ or } p < 1; \\ \text{any } 0 < \gamma < 1, & \text{if } \alpha \geqslant 1 \text{ and } p \geqslant 1; \end{cases}$$
$$\beta^* = \begin{cases} \min\{1, \beta, q\}, & \text{if } \beta < 1 \text{ or } q < 1; \\ \text{any } 0 < \gamma < 1, & \text{if } \beta \geqslant 1 \text{ and } q \geqslant 1. \end{cases}$$

Proof. It is just a combination of the arguments in [5, Theorem A.1, Corollary A.2] and [15, Section 5.1]. We stress that Proposition 2.1 is the key point which enables the bootstrap argument. □

Proof of Theorem 1.1 completed. This is a direct combination of Lemmas 2.6, 2.7 and 2.8. As for the statement concerning the radial monotonicity of u and v, we observe that the problem can be written as

$$-\left(r^{N-1}u'\right)' = r^{\beta+N-1}v^q, \qquad -\left(r^{N-1}v'\right)' = r^{\alpha+N-1}u^p,$$
$$u'(0) = v'(0) = u(1) = v(1) = 0;$$

since $u, v > 0$ we have that $u', v' < 0$ in $(0, 1)$. □

In view of deriving some estimates on the least energy radial solutions, it will be convenient to state an equivalent formulation of the problem (1.1). The starting point is the observation that the inequality (H1) holds if and only if it is possible to find $s > 1$ in such a way that the embeddings

$$W^{1,s}_{0,\mathrm{rad}}(B) \hookrightarrow L^{p+1}\left(B, |x|^\alpha\right) \quad \text{and} \quad W^{1,\frac{s}{s-1}}_{0,\mathrm{rad}}(B) \hookrightarrow L^{q+1}\left(B, |x|^\beta\right) \tag{2.10}$$

are compact. In this case, the functional $I_{s,\mathrm{rad}} : W^{1,s}_{0,\mathrm{rad}}(B) \times W^{1,\frac{s}{s-1}}_{0,\mathrm{rad}}(B) \to \mathbb{R}$ defined by

$$I_{s,\mathrm{rad}}(u, v) = \int_B \langle \nabla u, \nabla v \rangle\, dx - \int_B \left(\frac{1}{p+1}|u|^{p+1}|x|^\alpha + \frac{1}{q+1}|v|^{q+1}|x|^\beta \right) dx,$$

is a C^1-functional. Observe that if $(p+1)(N-2) < 2N + 2\alpha$ and $(q+1)(N-2) < 2N + 2\beta$, we can choose $s = 2$ and so $I_{s,\mathrm{rad}}$ is defined on $H^1_{0,\mathrm{rad}}(B) \times H^1_{0,\mathrm{rad}}(B)$.

If (u, v) is a critical point of $I_{s,\mathrm{rad}}$, that is, $(u, v) \in W^{1,s}_{0,\mathrm{rad}}(B) \times W^{1,\frac{s}{s-1}}_{0,\mathrm{rad}}(B)$ satisfies

$$\int_B \left(\langle \nabla u, \nabla \psi \rangle + \langle \nabla \varphi, \nabla v \rangle \right) dx = \int_B \left(|v|^{q-1}v\psi|x|^\beta + |u|^{p-1}u\varphi|x|^\alpha \right) dx,$$
$$\forall (\varphi, \psi) \in W^{1,s}_{0,\mathrm{rad}}(B) \times W^{1,\frac{s}{s-1}}_{0,\mathrm{rad}}(B),$$

then one easily proves that $(u, v) \in C^{2,\beta^*}(\overline{B}) \times C^{2,\alpha^*}(\overline{B})$ is a classical solution of (1.1) with

$$\alpha^* = \begin{cases} \min\{1, \alpha, p\}, & \text{if } \alpha < 1 \text{ or } p < 1; \\ \text{any } 0 < \gamma < 1, & \text{if } \alpha \geqslant 1 \text{ and } p \geqslant 1; \end{cases}$$
$$\beta^* = \begin{cases} \min\{1, \beta, q\}, & \text{if } \beta < 1 \text{ or } q < 1; \\ \text{any } 0 < \gamma < 1, & \text{if } \beta \geqslant 1 \text{ and } q \geqslant 1. \end{cases}$$

In addition, if $u \in E_{\mathrm{rad}}$ is a critical point of J_{rad} and $-v = |x|^{-\frac{\beta}{q}}|\Delta u|^{\frac{1}{q}-1}\Delta u$, then $(u, v) \in W^{1,s}_{0,\mathrm{rad}}(B) \times W^{1,\frac{s}{s-1}}_{0,\mathrm{rad}}(B)$ is a critical point of $I_{s,\mathrm{rad}}$ and $I_{s,\mathrm{rad}}(u, v) = J_{\mathrm{rad}}(u)$. Conversely, if $(u, v) \in W^{1,s}_{0,\mathrm{rad}}(B) \times W^{1,\frac{s}{s-1}}_{0,\mathrm{rad}}(B)$ is a critical point of $I_{s,\mathrm{rad}}$, then $u \in E_{\mathrm{rad}}$ is a critical point of J_{rad}. As a consequence,

$$c_{J_{\mathrm{rad}}} = \inf\bigl\{I_{s,\mathrm{rad}}(u, v)\colon\ (u, v) \text{ is a nonzero critical point of } I_{s,\mathrm{rad}}\bigr\}. \tag{2.11}$$

In virtue of this, for short, we write c_{rad} in place of $c_{J_{\mathrm{rad}}}$.

2.2. Asymptotics

We aim at giving a precise estimate on the growth of the radial ground critical level c_{rad} when say, β is fixed and $\alpha \to \infty$. More generally, we prove the following.

Proposition 2.9. *Assume* (H1) *and* $pq > 1$. *There exist* $a, b, \delta, \alpha_0 > 0$ *such that, for every* $\alpha \geqslant \alpha_0$, $0 \leqslant \beta \leqslant \delta\alpha$,

$$a\alpha^{\frac{(p+2)(q+1)}{pq-1}}\bigl(1 + \beta^{\frac{p+1}{pq-1}}\bigr) \leqslant c_{\mathrm{rad}} \leqslant b\alpha^{\frac{(p+2)(q+1)}{pq-1}}\bigl(1 + \beta^{\frac{p+1}{pq-1}}\bigr).$$

In the sequel we always assume that (H1) and $pq > 1$ hold. The first inequality in Proposition 2.9 is a direct consequence of the following.

Lemma 2.10. *There exists* $c_1 = c_1(p, q, N) > 0$ *such that for every* $\alpha, \beta \geqslant 0$, *one has*

$$c_{\mathrm{rad}} \geqslant c_1\left[(\alpha + N)^{q+1}(\beta + N)^{p+1}\left(\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} - (N-2)\right)^{(p+1)(q+1)}\right]^{\frac{1}{pq-1}}. \tag{2.12}$$

Proof. Let (u, v) be any nontrivial radially symmetric solution of (1.1), which we can assume to have least energy value. By taking (2.11) into account, we have that

$$c_{\mathrm{rad}} = \mu_{p,q}\int_B |u|^{p+1}|x|^\alpha\,dx$$

with $\mu_{p,q} = 1 - \frac{1}{p+1} - \frac{1}{q+1}$, which is positive since $pq > 1$. According to the well-known Pohožaev identity (see below)

$$\begin{aligned}\int_{\partial B}\langle\nabla u, \nabla v\rangle\,dS = {}&\int_B \frac{N+\alpha}{p+1}|u|^{p+1}|x|^\alpha\,dx + \int_B \frac{N+\beta}{q+1}|v|^{q+1}|x|^\beta\,dx \\ &- (N-2)\int_B\langle\nabla u, \nabla v\rangle\,dx,\end{aligned} \tag{2.13}$$

we have that

$$\int_{\partial B} \langle \nabla u, \nabla v \rangle \, dS = \left(\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} - (N-2) \right) \frac{c_{\mathrm{rad}}}{\mu_{p,q}}.$$

On the other hand, we can integrate the system in B and apply the Hölder inequality to deduce that

$$\left| \int_{\partial B} \langle \nabla u, x \rangle \, dS \right| \leqslant \frac{c_1}{(\beta+N)^{1/(q+1)}} c_{\mathrm{rad}}^{q/(q+1)},$$

$$\left| \int_{\partial B} \langle \nabla v, x \rangle \, dS \right| \leqslant \frac{c_1}{(\alpha+N)^{1/(p+1)}} c_{\mathrm{rad}}^{p/(p+1)},$$

for some $c_1 > 0$. Since u and v are radially symmetric, we have that

$$\omega_{N-1} \int_{\partial B} \langle \nabla u, \nabla v \rangle \, dS = \int_{\partial B} \langle \nabla u, x \rangle \, dS \int_{\partial B} \langle \nabla v, x \rangle \, dS,$$

where $\omega_{N-1} = |S^{N-1}|$, and the conclusion follows by combining these estimates. □

Remark 2.11. In the case $\alpha = \beta$ and $p = q$, this provides an alternative to the proof of the corresponding radial estimate in [30, Theorem 4.1], namely, $c_{\mathrm{rad}} \geqslant c\alpha^{\frac{p+3}{p-1}}$. On the other hand, in contrast with [30], our argument is also valid in the one-dimensional case $N = 1$.

The reversed inequality in Proposition 2.9 is more delicate to establish. We first prove some auxiliary results which will be used both in the proof of Proposition 2.9 and in Section 3.

In order to emphasize the dependence on α and β, let us denote by $c_{\alpha,\beta}$ the (up to a multiplicative factor) radially symmetric ground level of the system, namely

$$c_{\alpha,\beta} = \int_B |x|^\alpha u_{\alpha,\beta}^{p+1} \, dx = \int_B |x|^\beta v_{\alpha,\beta}^{q+1} \, dx = \int_B \langle \nabla u_{\alpha,\beta}, \nabla v_{\alpha,\beta} \rangle \, dx,$$

where (u_α, v_α) is a least energy radially symmetric solution of the system, which in this case can also be written as

$$-\left(r^{N-1} u'\right)' = r^{\beta+N-1} v^q, \qquad -\left(r^{N-1} v'\right)' = r^{\alpha+N-1} u^p,$$
$$u'(0) = v'(0) = u(1) = v(1) = 0.$$

Since $u, v > 0$, we have $u', v' < 0$ in $(0, 1)$. In order to simplify the notations, we occasionally drop the subscripts α, β in $(u_{\alpha,\beta}, v_{\alpha,\beta})$.

A basic tool we use is the Pohožaev identity in a form that we now recall. Denote by $W(x)$ the vector field

$$W(x) = \langle \nabla v, x \rangle \nabla u + \langle \nabla u, x \rangle \nabla v - \langle \nabla u, \nabla v \rangle x + |x|^\alpha \frac{u^{p+1}}{p+1} x + |x|^\beta \frac{v^{q+1}}{q+1} x,$$

so that

$$\operatorname{div} W = (2-N)\langle \nabla u, \nabla v \rangle + \frac{\alpha + N}{p+1}|x|^\alpha u^{p+1} + \frac{\beta + N}{q+1}|x|^\beta v^{q+1}.$$

Take $\gamma \geqslant 0$. Since $\langle \nabla u, x \rangle \langle \nabla v, x \rangle = \langle \nabla u, \nabla v \rangle |x|^2$ we have

$$\begin{aligned} \operatorname{div}\bigl(W|x|^{-\gamma}\bigr) &= |x|^{-\gamma} \operatorname{div} W + \bigl\langle W, \nabla |x|^{-\gamma} \bigr\rangle \\ &= (2-N-\gamma)|x|^{-\gamma} \langle \nabla u, \nabla v \rangle + \frac{\alpha + N - \gamma}{p+1}|x|^{\alpha-\gamma} u^{p+1} \\ &\quad + \frac{\beta + N - \gamma}{q+1}|x|^{\beta - \gamma} v^{q+1}. \end{aligned}$$

Lemma 2.12. *There exists $C_0 \in \mathbb{R}$ independent of α, β such that*

$$\bigl|S^{N-1}\bigr| u'_{\alpha,\beta}(1) v'_{\alpha,\beta}(1) = \int_{\partial B} \langle \nabla u_{\alpha,\beta}, \nabla v_{\alpha,\beta} \rangle \, dS = \left(\frac{\alpha}{p+1} + \frac{\beta}{q+1} + C_0 \right) c_{\alpha,\beta}.$$

Proof. This follows at once by integrating in B the Pohožaev identity with $\gamma = 0$, as in (2.13). □

Lemma 2.13. *Given $\alpha > \beta + 1$, we have that*

$$\int_B |x|^{-\beta - N} \langle \nabla u_{\alpha,\beta}, \nabla v_{\alpha,\beta} \rangle \, dx \leqslant \int_B |x|^{\alpha - \beta - N} u_{\alpha,\beta}^{p+1} \, dx.$$

Proof. Let $\gamma = \beta + N$, so that $\alpha > \gamma - N + 1$. It follows from the equation $-(r^{N-1}v')' = r^{\alpha+N-1}u^p$ that

$$-r^{N-1} v'(r) = \int_0^r s^{\alpha + N - 1} u^p(s) \, ds \leqslant r^{\alpha + N - 1} \int_0^1 u^p(s) \, ds,$$

and therefore, whatever $r \in [0, 1]$, we have

$$\bigl|v'(r)\bigr| \leqslant r^\alpha \int_0^1 u^p(s) \, ds. \tag{2.14}$$

We now multiply the equation by $r^{-\gamma} u(r)$ and use (2.14) to integrate by parts, so as to conclude that

$$\begin{aligned}\int_0^1 r^{\alpha-\gamma+N-1}u^{p+1}\,dr &= \int_0^1 r^{N-1}v'\big(r^{-\gamma}u\big)'\,dr\\ &= \int_0^1 r^{N+1-\gamma}u'v'\,dr - \gamma\int_0^1 r^{N-2-\gamma}uv'\,dr\\ &\geqslant \int_0^1 r^{N+1-\gamma}u'v'\,dr,\end{aligned}$$

as intended. □

Lemma 2.14. *Assume* $0<\delta<1/(p+2)$. *There exist* $C_1,\alpha_0>0$ *such that, for every* $\alpha\geqslant\alpha_0$, $0\leqslant\beta\leqslant\delta\alpha$, *we have*

$$\int_B |x|^{\alpha-\beta-N}u_{\alpha,\beta}^{p+1}\,dx\leqslant C_1c_{\alpha,\beta}. \tag{2.15}$$

Proof. Given $0\leqslant\gamma\leqslant\beta+N$, it follows from the Pohožaev identity that

$$\operatorname{div}\big(W|x|^{-\gamma}\big)\geqslant(2-N-\gamma)|x|^{-\gamma}\langle\nabla u,\nabla v\rangle+\frac{\alpha+N-\gamma}{p+1}|x|^{\alpha-\gamma}u^{p+1}$$

and so

$$\frac{\alpha+N-\gamma}{p+1}\int_B|x|^{\alpha-\gamma}u^{p+1}\,dx\leqslant(N+\gamma-2)\int_B|x|^{-\gamma}\langle\nabla u,\nabla v\rangle\,dx+\int_{\partial B}\langle\nabla u,\nabla v\rangle\,dS.$$

We choose $\gamma=\beta+N$. Then, since $\alpha>\beta+1$ for large values of α, Lemma 2.13 implies that

$$\frac{\alpha-\beta}{p+1}\int_B|x|^{\alpha-\beta-N}u^{p+1}\,dx\leqslant(2N-2+\beta)\int_B|x|^{\alpha-\beta-N}u^{p+1}\,dx+\int_{\partial B}\langle\nabla u,\nabla v\rangle\,dS.$$

The conclusion follows by using Lemma 2.12 and by recalling that $\beta\leqslant\delta\alpha$ with $\delta<1/(p+2)$. □

Lemma 2.15. *Assume* $0<\delta<1/(p+2)$. *There exist* $C_2,\alpha_0>0$ *such that, for every* $\alpha\geqslant\alpha_0$, $0\leqslant\beta\leqslant\delta\alpha$,

$$u_{\alpha,\beta}(0)\leqslant C_2\bigg(\frac{1}{1+\beta^{(q+2)/(q+1)}}\bigg)c_{\alpha,\beta}^{\frac{q}{q+1}}$$

and

$$v_{\alpha,\beta}(0)\leqslant C_2\big(1+\beta^{\frac{1}{q+1}}\big)c_{\alpha,\beta}^{\frac{1}{q+1}}.$$

In particular, for some $C_3 > 0$,

$$u_{\alpha,\beta}(0)v_{\alpha,\beta}^{q+2}(0) \leqslant C_3 c_{\alpha,\beta}^2, \quad \forall \alpha \geqslant \alpha_0,\ 0 \leqslant \beta \leqslant \delta\alpha. \tag{2.16}$$

Proof. We multiply the first ODE of our system by $r^{1-N-\beta}v'$ and the second one by $r^{1-N-\beta}u'$. Thanks to (2.14) we can integrate by parts to deduce that

$$\int_0^1 r^{\alpha-\beta}u^p u'\,dr + \int_0^1 v^q v'\,dr = -u'(1)v'(1) - (2N+\beta-2)\int_0^1 u'v'r^{-\beta-1}\,dr,$$

that is,

$$\frac{\alpha-\beta}{p+1}\int_0^1 r^{\alpha-\beta-1}u^{p+1}\,dr + \frac{1}{q+1}v^{q+1}(0) = u'(1)v'(1) + (2N+\beta-2)\int_0^1 u'v'r^{-\beta-1}\,dr.$$

Since $r^{\alpha+N-1} \leqslant r^{\alpha-\beta-1}$, we conclude that there exist $a, b > 0$ such that

$$\frac{\alpha-\beta}{p+1}c_{\alpha,\beta} + \left|S^{N-1}\right|\frac{1}{q+1}v^{q+1}(0) \leqslant \left|S^{N-1}\right|u'(1)v'(1) + (a+b\beta)\int_B |x|^{-\beta-N}\langle\nabla u, \nabla v\rangle\,dx.$$

By combining Lemmas 2.13 and 2.14, this implies that there exist $a, b > 0$ such that

$$\frac{\alpha-\beta}{p+1}c_{\alpha,\beta} + \left|S^{N-1}\right|\frac{1}{q+1}v^{q+1}(0) \leqslant \left|S^{N-1}\right|u'(1)v'(1) + (a+b\beta)c_{\alpha,\beta}.$$

Using Lemma 2.12 we conclude that

$$v^{q+1}(0) \leqslant C_2(1+\beta)c_{\alpha,\beta},$$

as intended. As for $u(0)$, we have that $-(r^{N-1}u')' = r^{\beta+N-1}v^q(r) \leqslant r^{\beta+N-1}v^q(0)$. Integrating twice this last inequality, we deduce that

$$u(0) \leqslant \frac{v^q(0)}{(\beta+2)(\beta+N)}$$

and the conclusion follows easily. $\square$

Lemma 2.16. *There exist $b, \delta, \alpha_0 > 0$ such that, for every $\alpha \geqslant \alpha_0$, $0 \leqslant \beta \leqslant \delta\alpha$,*

$$c_{\alpha,\beta} \leqslant b\alpha^{\frac{(p+2)(q+1)}{pq-1}}\left(1 + \beta^{\frac{p+1}{pq-1}}\right).$$

Proof. We first observe that since $c_{\alpha,\beta}$ increases with β (as follows directly from (2.8)), we may assume that $\beta \geqslant 1$. We multiply the equation $-\Delta u = |x|^\beta v^q$ by $|x|^\alpha$ and integrate twice by parts. Then

$$\begin{aligned}
-\int_{\partial B} \frac{\partial u}{\partial n}\, dS &= \alpha(\alpha+N-2)\int_B |x|^{\alpha-2} u\, dx + \int_B |x|^\beta v^q |x|^\alpha\, dx \\
&\leqslant 2\alpha^2 \Big(\int_B |x|^{\alpha-2} u^{p+1}\, dx\Big)^{1/(p+1)} \Big(\int_B |x|^{\alpha-2}\, dx\Big)^{p/(p+1)} \\
&\quad + \Big(\int_B |x|^\beta v^{q+1}\, dx\Big)^{q/(q+1)} \Big(\int_B |x|^\beta |x|^{\alpha(q+1)}\, dx\Big)^{1/(q+1)}
\end{aligned}$$

and so, thanks to (2.15) (observe that $\alpha - 2 \geqslant \alpha - \beta - N$),

$$\int_{\partial B} \left|\frac{\partial u}{\partial n}\right| dS \leqslant C\big(c_{\alpha,\beta}^{\frac{1}{p+1}} \alpha^{\frac{p+2}{p+1}} + c_{\alpha,\beta}^{\frac{q}{q+1}} \alpha^{\frac{-1}{q+1}}\big). \tag{2.17}$$

On the other hand, by using the equation $-\Delta u = |x|^\beta v^q$ together with Lemma 2.15 we see that

$$\begin{aligned}
c_{\alpha,\beta} &= \int_B |x|^\beta v^{q+1}\, dx \leqslant v(0) \int_B |x|^\beta v^q\, dx \\
&= v(0) \int_{\partial B} \left|\frac{\partial u}{\partial n}\right| dS \leqslant C' \beta^{\frac{1}{q+1}} c_{\alpha,\beta}^{\frac{1}{q+1}} \int_{\partial B} \left|\frac{\partial u}{\partial n}\right| dS.
\end{aligned} \tag{2.18}$$

By combining (2.17) and (2.18), we obtain that

$$c_{\alpha,\beta} \leqslant CC' \bigg(c_{\alpha,\beta}^{\frac{p+q+2}{(p+1)(q+1)}} \beta^{\frac{1}{q+1}} \alpha^{\frac{p+2}{p+1}} + c_{\alpha,\beta} \Big(\frac{\beta}{\alpha}\Big)^{\frac{1}{q+1}} \bigg),$$

and the conclusion follows. □

Proof of Proposition 2.9 completed. This is a consequence of Lemmas 2.10 and 2.16. □

In the remaining of this section, we establish additional estimates which will be used in Section 3.

Lemma 2.17. *Assume $0 < \delta < 1/(p+2)$ and $N > 2$. There exist $C_4, \alpha_0 > 0$ such that, for every $\alpha \geqslant \alpha_0$, $0 \leqslant \beta \leqslant \delta\alpha$,*

$$\Big(\int_B |\nabla u_{\alpha,\beta}|^2\, dx\Big)^{1/2} \Big(\int_B \frac{v_{\alpha,\beta}^2}{|x|^2}\, dx\Big)^{1/2} \leqslant \frac{C_4}{1+\sqrt{\beta}} c_{\alpha,\beta}.$$

Proof. We have that $\int_B v^2/|x|^2\,dx \leqslant v^2(0)\int_B (1/|x|^2)\,dx$ and

$$\int_B |\nabla u|^2\,dx = \int_B |x|^\beta v^q u\,dx \leqslant v^q(0)u(0)\int_B |x|^\beta\,dx = \left|S^{N-1}\right|\frac{v^q(0)u(0)}{N+\beta},$$

and so, thanks also to (2.16),

$$\int_B |\nabla u|^2\,dx \int_B \frac{v^2}{|x|^2}\,dx \leqslant Cu(0)v^{q+2}(0)\frac{1}{N+\beta} \leqslant C' c_{\alpha,\beta}^2 \frac{1}{N+\beta},$$

as claimed. □

For positive real sequences (x_α), (y_α), we use the notation $x_\alpha \sim y_\alpha$ with the meaning that there exist $a, b, \alpha_0 > 0$ such that

$$a \leqslant \frac{x_\alpha}{y_\alpha} \leqslant b, \quad \forall \alpha \geqslant \alpha_0.$$

As we will freeze β, we omit the subscript β in the notations $u_{\alpha,\beta}$, $v_{\alpha,\beta}$, $c_{\alpha,\beta}$.

Lemma 2.18. *Let $\beta \geqslant 0$. Then, as $\alpha \to \infty$, there holds*

$$v_\alpha(0) \sim c_\alpha^{\frac{1}{q+1}}, \qquad u_\alpha(0) \sim c_\alpha^{\frac{q}{q+1}}, \qquad \left|v_\alpha'(1)\right| \sim \alpha c_\alpha^{\frac{1}{q+1}}, \qquad \left|u_\alpha'(1)\right| \sim c_\alpha^{\frac{q}{q+1}}.$$

Moreover, we have

$$\int_B |\nabla v_\alpha|\,dx \sim c_\alpha^{\frac{1}{q+1}} \quad \text{and} \quad \left(\int_B |\nabla u_\alpha|^2\,dx\right)^{1/2} \sim c_\alpha^{\frac{q}{q+1}}.$$

Proof. As in the proof of (2.12), it follows from the Hölder inequality that $|v_\alpha'(1)| \leqslant C\frac{c_\alpha^{\frac{p}{p+1}}}{\alpha^{\frac{1}{p+1}}} \sim \alpha c_\alpha^{\frac{1}{q+1}}$ and $|u_\alpha'(1)| \leqslant C c_\alpha^{\frac{q}{q+1}}$. Since, by Proposition 2.9 and Lemma 2.12, $\frac{c_\alpha^{\frac{p}{p+1}}}{\alpha^{\frac{1}{p+1}}} c_\alpha^{\frac{q}{q+1}} \sim \alpha c_\alpha \sim u_\alpha'(1)v_\alpha'(1)$, this yields the asymptotic estimates for both $u_\alpha'(1)$ and $v_\alpha'(1)$. Also, we see from (2.18) and Lemma 2.15 that $c_\alpha \leqslant C v_\alpha(0)|u_\alpha'(1)| \leqslant C' c_\alpha^{\frac{1}{q+1}} c_\alpha^{\frac{q}{q+1}} = C' c_\alpha$, and so $v_\alpha(0) \sim c_\alpha^{\frac{1}{q+1}}$.

On the other hand, by using the Hölder inequality we have that

$$r^{N-1}\left|v_\alpha'(r)\right| = \int_0^r s^\alpha u_\alpha^p(s) s^{N-1}\,ds \leqslant C c_\alpha^{p/(p+1)} \alpha^{-1/(p+1)} r^{(\alpha+N)/(p+1)}; \tag{2.19}$$

by integrating over the interval [0, 1], we get that

$$\int_B |\nabla v_\alpha|\,dx \leqslant C' c_\alpha^{\frac{p}{p+1}} \alpha^{-\frac{p+2}{p+1}} \sim c_\alpha^{\frac{1}{q+1}}.$$

As for the reversed inequality, it is enough to observe that

$$c_\alpha = \int_B \langle \nabla u_\alpha, \nabla v_\alpha \rangle \, dx \leqslant \left| u'_\alpha(1) \right| \int_B |\nabla v_\alpha| \, dx \leqslant C c_\alpha^{\frac{q}{q+1}} c_\alpha^{\frac{1}{q+1}} = C c_\alpha.$$

Finally, we have already proved that $u_\alpha(0) \leqslant C c_\alpha^{\frac{q}{q+1}}$ (cf. Lemma 2.15) and $\int_B |\nabla u_\alpha|^2 \, dx \leqslant C c_\alpha^{\frac{2q}{q+1}}$ (cf. the proof of Lemma 2.17). Take

$$z_\alpha = \frac{u_\alpha}{c_\alpha^{q/(q+1)}} \quad \text{and} \quad w_\alpha = \frac{v_\alpha}{c_\alpha^{1/(q+1)}},$$

so that

$$-\Delta z_\alpha = |x|^\beta w_\alpha^q. \tag{2.20}$$

We have that (z_α) is bounded in $C^1(B) \cap H_0^1(B)$ and (w_α) is bounded in $L^\infty(B) \cap W^{1,1}(B)$. By applying elliptic regularity to Eq. (2.20), we get that, up to a subsequence, $z_\alpha \to z$ in $C^1(B)$. Moreover, we have $z \neq 0$, since $|u'_\alpha(1)| \sim c_\alpha^{\frac{q}{q+1}}$. This implies that $\int_B |\nabla u_\alpha|^2 \, dx \sim c_\alpha^{\frac{2q}{q+1}}$ and $u_\alpha(0) \sim c_\alpha^{\frac{q}{q+1}}$.

Incidentally, since $w_\alpha \to w$ in every $L^s(B)$ with $1 \leqslant s < \infty$ and, by taking (2.19) into account, we see that w is a constant function and $w_\alpha \to w$ uniformly in compact subsets of $[0, 1)$. Passing to the limit in (2.20) we deduce that $-\Delta z = |x|^\beta w^q$, and so $w \neq 0$; thus w_α exhibits a Dirac type behavior on the boundary of the set B. $\square$

3. Ground state solutions

3.1. Existence and regularity

In the absence of a regularity result similar to the one in Proposition 2.1 we analyze carefully the properties of the underlying Sobolev space associated to (1.1). Throughout this section, we always assume that (H2) holds. Let us consider the space

$$E = \left\{ u \in W^{2,\frac{q+1}{q}}(B) \cap W_0^{1,\frac{q+1}{q}}(B): \int_B |\Delta u|^{\frac{q+1}{q}} |x|^{-\frac{\beta}{q}} \, dx < +\infty \right\} \tag{3.1}$$

endowed with the norm

$$\|u\| = \left(\int_B |\Delta u|^{\frac{q+1}{q}} |x|^{-\frac{\beta}{q}} \, dx \right)^{q/(q+1)}.$$

Then E is a Banach space and it is compactly imbedded in $L^{p+1}(B, |x|^\alpha)$; we recall that (H2) is the precise condition for having a compact imbedding of $W^{2,\frac{q+1}{q}}(B) \cap W_0^{1,\frac{p+1}{p}}(B)$ into $L^{p+1}(B)$. The following three results are easily proved.

Lemma 3.1. $-\Delta : E \to L^{\frac{q+1}{q}}(B, |x|^{-\beta/q})$ *is an isometric isomorphism.*

Lemma 3.2. *The map* $S : L^{\frac{q+1}{q}}(B) \to L^{\frac{q+1}{q}}(B, |x|^{-\beta/q})$ *defined by* $S(u) = u|x|^{\beta/(q+1)}$*, is an isometric isomorphism. In particular,* $D = \{f \in L^{\frac{q+1}{q}}(B, |x|^{-\beta/q})\colon\ f|x|^{-\beta/(q+1)} \in C^{0,\gamma}(\overline{B})\} \subset C^{0,\gamma}(\overline{B})$ *is a dense subspace in* $L^{\frac{q+1}{q}}(B, |x|^{-\beta/q})$*, with* $\gamma = \min\{1, \frac{\beta}{q+1}\}$.

Lemma 3.3. *The space* $D' = \{u \in C^{2,\gamma}(\overline{B})\colon\ u = 0$ *on* ∂B *and* $\Delta u(x)|x|^{-\beta/(q+1)} \in C^{0,\gamma}(\overline{B})\}$ *is a dense subspace in* E.

Moreover, arguing as in [17, Lemma 3.2], we can prove the following Riesz representation theorem for the dual space E'.

Lemma 3.4. *For each* $\Phi \in E'$ *there exists a unique* $u \in E$ *such that*

$$\langle \Phi, \varphi \rangle = \int_B |\Delta u|^{\frac{1}{q}-1} \Delta u \Delta \varphi |x|^{-\beta/q}\, dx, \quad \forall \varphi \in E.$$

We will use the (nonlinear) map $T : E \to E'$ given by

$$\bigl\langle T(u), \varphi \bigr\rangle = \int_B |\Delta u|^{\frac{1}{q}-1} \Delta u \Delta \varphi |x|^{-\beta/q}\, dx, \quad \forall u, \varphi \in E. \tag{3.2}$$

For every $w \in L^{\frac{p+1}{p}}(B)$, the imbedding $E \hookrightarrow L^{p+1}(B)$ guarantees that the map

$$\varphi \mapsto \int_B w\varphi\, dx, \quad \varphi \in E,$$

defines a continuous linear functional on E and so, by Lemma 3.4, there exists a unique $u \in E$ such that $T(u) = w$, that is

$$\int_B |\Delta u|^{\frac{1}{q}-1} \Delta u \Delta \varphi |x|^{-\beta/q}\, dx = \int_B w\varphi\, dx, \quad \forall \varphi \in E. \tag{3.3}$$

Lemma 3.5. *Let* $w \in L^{\frac{p+1}{p}}(B)$ *and* $u \in E$ *be such that* $T(u) = w$*. Set* $-v = |x|^{-\beta/q}|\Delta u|^{\frac{1}{q}-1}\Delta u$*. Then* $v \in W^{2,\frac{p+1}{p}}(B) \cap W_0^{1,\frac{p+1}{p}}(B)$ *and* u *and* v *are strong solutions of*

$$\begin{cases} -\Delta u = |x|^\beta |v|^{q-1} v, \quad -\Delta v = w & \text{in } B,\\ u, v = 0 & \text{on } \partial B. \end{cases}$$

Proof. Let v_0 and z be the strong solutions of

$$\begin{cases} -\Delta v_0 = w & \text{in } B,\\ v_0 = 0 & \text{on } \partial B, \end{cases} \qquad \begin{cases} -\Delta z = |x|^\beta |v_0|^{q-1} v_0 & \text{in } B,\\ z = 0 & \text{on } \partial B. \end{cases}$$

Then $z \in E$ and, using (3.3),

$$\int_B |\Delta u|^{\frac{1}{q}-1}\Delta u\Delta\varphi|x|^{-\frac{\beta}{q}}\,dx = \int_B w\varphi\,dx = \int_B (-\Delta v_0)\varphi\,dx = \int_B v_0(-\Delta\varphi)\,dx$$
$$= \int_B |\Delta z|^{\frac{1}{q}-1}\Delta z\Delta\varphi|x|^{-\frac{\beta}{q}}\,dx$$

for all $\varphi \in D'$. Hence, from Lemmas 3.3 and 3.4, $z = u$. As a consequence, also $v_0 = v$. □

Similarly to Section 2, we say that u is a weak solution of

$$\Delta\big(|x|^{-\frac{\beta}{q}}|\Delta u|^{\frac{1}{q}-1}\Delta u\big) = |x|^{\alpha}|u|^{p-1}u \quad \text{in } B, \text{ with } u, \Delta u = 0 \text{ on } \partial B, \tag{3.4}$$

if u is a critical point of the $C^1(E,\mathbb{R})$-functional

$$J(u) = \frac{q}{q+1}\int_B |\Delta u|^{\frac{q+1}{q}}|x|^{-\frac{\beta}{q}}\,dx - \frac{1}{p+1}\int_B |u|^{p+1}|x|^{\alpha}\,dx, \quad u \in E,$$

that is, $u \in E$ satisfies

$$\int_B |\Delta u|^{\frac{1}{q}-1}\Delta u\Delta\varphi|x|^{-\frac{\beta}{q}}\,dx = \int_B |u|^{p-1}u\varphi|x|^{\alpha}\,dx, \quad \forall\varphi \in E.$$

Moreover, a function $u \in E\backslash\{0\}$ is said to be a *ground state solution* for (1.1) if J attains its smallest nonzero critical value at u.

Lemma 3.6. *Assume* (H2). *Let $u \in E$ be a weak solution of* (2.4) *and set $-v = |x|^{-\frac{\beta}{q}}|\Delta u|^{\frac{1}{q}-1}\Delta u$. Then $(u, v) \in C^{2,\beta^*}(\overline{B}) \times C^{2,\alpha^*}(\overline{B})$ is a classical solution of* (1.1) *with*

$$\alpha^* = \begin{cases} \min\{1,\alpha,p\}, & \textit{if } \alpha < 1 \textit{ or } p < 1; \\ \textit{any } 0 < \gamma < 1, & \textit{if } \alpha \geqslant 1 \textit{ and } p \geqslant 1; \end{cases}$$
$$\beta^* = \begin{cases} \min\{1,\beta,q\}, & \textit{if } \beta < 1 \textit{ or } q < 1; \\ \textit{any } 0 < \gamma < 1, & \textit{if } \beta \geqslant 1 \textit{ and } q \geqslant 1. \end{cases}$$

Proof. Let u be a weak solution of (2.4); by definition, we have that $|x|^{\alpha}|u|^{q-1}u = T(u)$. Then, by Lemma 3.5 with $-v = |x|^{-\beta/q}|\Delta u|^{\frac{1}{q}-1}\Delta u$, we have that (u, v) is a strong solution of (1.1) such that $u \in W^{2,\frac{q+1}{q}}(B)\cap W_0^{1,\frac{q+1}{q}}(B)$ and $v \in W^{2,\frac{p+1}{p}}(B)\cap W_0^{1,\frac{p+1}{p}}(B)$. Using (H2) and arguing as in the proofs of [17, Theorem 1.1] and [5, Theorem 1.7], $(u, v) \in C^{2,\beta^*}(\overline{B}) \times C^{2,\alpha^*}(\overline{B})$ with α^* and β^* as in the statement. □

Proof of Theorem 1.2 completed. In view of Lemma 3.6, we can proceed as in the proof of [5, Theorem 1.4]. Again, a key point in our argument is the fact that if $u \in E$ then w, the strong solution of $-\Delta w = |\Delta u|$ in B with $w = 0$ on ∂B, also lies in E. □

We would like to mention that, similarly to Section 2, an equivalent (but sometimes less convenient to work with) formulation of the problem can be described as follows. A necessary and sufficient condition for (H2) to hold is the possibility of finding $s > 1$ in such a way that the embeddings

$$W_0^{1,s}(B) \hookrightarrow L^{p+1}\big(B, |x|^\alpha\big) \quad \text{and} \quad W_0^{1,\frac{s}{s-1}}(B) \hookrightarrow L^{q+1}\big(B, |x|^\beta\big) \tag{3.5}$$

are compact. In this case, the functional $I_s : W_0^{1,s}(B) \times W_0^{1,\frac{s}{s-1}}(B) \to \mathbb{R}$ defined by

$$I_s(u,v) = \int_B \langle \nabla u, \nabla v\rangle\, dx - \int_B \left(\frac{1}{p+1}|u|^{p+1}|x|^\alpha + \frac{1}{q+1}|v|^{q+1}|x|^\beta\right) dx,$$

is a C^1-functional. We say that $(u,v) \in W_0^{1,s}(B) \times W_0^{1,\frac{s}{s-1}}(B)$ is a weak solution of (1.1) if (u,v) is a critical point of I_s, that is, $(u,v) \in W_0^{1,s}(B) \times W_0^{1,\frac{s}{s-1}}(B)$ satisfies

$$\int_B \big(\langle \nabla u, \nabla \psi\rangle + \langle \nabla \varphi, \nabla v\rangle\big)\, dx = \int_B \big(|v|^{q-1} v\psi |x|^\beta + |u|^{p-1}u\varphi|x|^\alpha\big)\, dx,$$
$$\forall (\varphi, \psi) \in W_0^{1,s}(B) \times W_0^{1,\frac{s}{s-1}}(B).$$

A regularity result similar to the one in Lemma 3.6 also holds in the present context. Eventually this leads to the conclusion that if $pq \neq 1$ then the ground critical level c_J of the functional J is also given by

$$c_J = \inf\big\{I_s(u,v)\colon\ (u,v) \text{ is a nonzero critical point of } I_s\big\}.$$

3.2. Foliated Schwarz symmetry

We consider the set $\mathcal{H}$ of all closed half-spaces H in $\mathbb{R}^N$ such that $0 \in \partial H$. For $H \in \mathcal{H}$, we denote by $\sigma_H : \mathbb{R}^N \to \mathbb{R}^N$ the reflection with respect to the boundary ∂H of H. For simplicity, we also put $\bar{x} = \sigma_H(x)$ for $x \in \mathbb{R}^N$ when the underlying half-space H is understood. For a measurable function $w : \mathbb{R}^N \to \mathbb{R}$ we define the polarization w_H of w relative to H by

$$w_H(x) = \begin{cases} \max\{w(x), w(\bar{x})\}, & x \in H, \\ \min\{w(x), w(\bar{x})\}, & x \in \mathbb{R}^N \setminus H. \end{cases}$$

We also denote $\overline{w}(x) := w(\bar{x})$. The following result is essentially due to [6] but we provide a rather elementary proof.

Lemma 3.7. *Let $f \in L^t(B)$, $1 < t < \infty$, and $H \in \mathcal{H}$. Let u and v be the strong solutions of*

$$\begin{cases} -\Delta u = f, \quad -\Delta v = f_H & \text{in } B, \\ u, v = 0 \quad \text{on } \partial B. \end{cases}$$

Then $v = v_H$ in B and $v \geqslant u_H$ in $H \cap B$. Moreover,

$$\int_B u\varphi\,dx \leqslant \int_B v\varphi_H\,dx, \quad \forall\varphi \in L^\infty(B). \tag{3.6}$$

In particular, if $f \geqslant 0$,

$$\int_B u^s\varphi\,dx \leqslant \int_B v^s\varphi_H\,dx, \quad \forall\varphi \in L^\infty(B),\ \varphi \geqslant 0,\ s > 1. \tag{3.7}$$

Proof. Without loss of generality, we can assume that f is smooth. Since $-\Delta(v - \bar{v}) = f_H - \overline{f_H} \geqslant 0$ in $H \cap B$ we deduce from the maximum principle that $v \geqslant \bar{v}$ in $H \cap B$. Similarly, $v \leqslant \bar{v}$ in $(\mathbb{R}^N \backslash H) \cap B$, and so $v = v_H$ in B. On the other hand, since, by definition, $f - f_H = \overline{f_H} - \overline{f}$, we have that $-\Delta(\bar{u} + u - \bar{v} - v) = \overline{f} + f - f_H - \overline{f_H} = 0$. It follows that $\bar{u} + u = \bar{v} + v$ in B; in particular, $u = v$ on $\partial H \cap B$. Then, by observing that $-\Delta(v - u) = f_H - f \geqslant 0$ in $H \cap B$ and $-\Delta(v - \bar{u}) = f_H - \overline{f} \geqslant 0$ in $H \cap B$, we conclude that $v \geqslant u$ in $H \cap B$ and $v \geqslant \bar{u}$ in $H \cap B$, that is $v \geqslant u_H$ in $H \cap B$.

Now, given $\varphi \in L^\infty(B)$, we must derive the inequality

$$\int_B u\varphi\,dx = \int_{H\cap B} (u\varphi + \bar{u}\bar{\varphi})\,dx \leqslant \int_{H\cap B} (v\varphi_H + \bar{v}\overline{\varphi_H})\,dx = \int_B v\varphi_H\,dx.$$

By replacing $\bar{v} = u + \bar{u} - v$ and $\overline{\varphi_H} = \varphi + \bar{\varphi} - \varphi_H$ in the above expression, we find that the inequality reads

$$\int_{H\cap B} \big[(\varphi_H - \varphi)(v - \bar{u}) + (\varphi_H - \bar{\varphi})(v - u)\big]\,dx \geqslant 0.$$

Clearly, this holds true since each of the four terms in parenthesis is non-negative, and this establishes (3.6).

Finally, in case $f \geqslant 0$, since moreover, $v = v_H$, the property (3.7) is a consequence of (3.6), as follows from [6, Lemma 9.1] applied to the map $j(r) = r^s$. □

Let T be the operator given by (3.2). The next lemma will allow to complete the proof of Theorem 1.3.

Lemma 3.8. *Let $w \in L^{\frac{p+1}{p}}(B)$ be non-negative and $u, \widetilde{u} \in E$ be such that $T(u) = w$ and $T(\widetilde{u}) = w_H$. Then $\langle T(u), u\rangle \leqslant \langle T(\widetilde{u}), \widetilde{u}\rangle$.*

Proof. Let v and $\widetilde{v}$ be the strong solutions of

$$-\Delta v = w, \qquad -\Delta\widetilde{v} = w_H \quad \text{in } B, \qquad v, \widetilde{v} = 0 \quad \text{on } \partial B.$$

Then, by Lemma 3.5, u and $\widetilde{u}$ are the strong solutions of

$$-\Delta u = |x|^\beta v^q, \qquad -\Delta \widetilde{u} = |x|^\beta (\widetilde{v})^q \quad \text{in } B, \qquad u, \widetilde{u} = 0 \quad \text{on } \partial B$$

and, by definition,

$$\langle T(u), u\rangle = \int_B |\Delta u|^{\frac{q+1}{q}} |x|^{-\beta/q}\,dx = \int_B |x|^\beta v^{q+1}\,dx \quad \text{and} \quad \langle T(\widetilde{u}), \widetilde{u}\rangle = \int_B |x|^\beta (\widetilde{v})^{q+1}\,dx.$$

The conclusion follows then from (3.7) with $\varphi(x) = |x|^\beta$. $\square$

Proof of Theorem 1.3 completed. Let

$$S := \inf_{u \in E,\, u \neq 0} \frac{\langle T(u), u\rangle}{(\int_B |u|^{p+1}|x|^\alpha\,dx)^{(q+1)/q(p+1)}}.$$

Let S be achieved by a (positive) function u such that $\int_B u^{p+1}|x|^\alpha\,dx = 1$. Then $T(u) = S|x|^\alpha u^p$. Let $\widetilde{u}$ be such that $T(\widetilde{u}) = S|x|^\alpha u_H^p$. Then, by Lemma 3.8, $\langle T(u), u\rangle \leqslant \langle T(\widetilde{u}), \widetilde{u}\rangle$. By using the Hölder inequality we deduce that

$$\begin{aligned}
S = S\int_B |u|^{p+1}|x|^\alpha\,dx = \langle T(u), u\rangle \leqslant \langle T(\widetilde{u}), \widetilde{u}\rangle = S\int_B \widetilde{u} u_H^p |x|^\alpha\,dx \\
\leqslant S\Big(\int_B \widetilde{u}^{p+1}|x|^\alpha\,dx\Big)^{1/(p+1)}\Big(\int_B u_H^{p+1}|x|^\alpha\,dx\Big)^{p/(p+1)} = S\Big(\int_B \widetilde{u}^{p+1}|x|^\alpha\,dx\Big)^{1/(p+1)}
\end{aligned}$$

yielding that $\int_B \widetilde{u}^{p+1}|x|^\alpha\,dx \geqslant 1$ and so

$$\begin{aligned}
S &\leqslant \frac{\langle T(\widetilde{u}), \widetilde{u}\rangle}{(\int_B |\widetilde{u}|^{p+1}|x|^\alpha\,dx)^{(q+1)/q(p+1)}} \\
&\leqslant S\frac{(\int_B \widetilde{u}^{p+1}|x|^\alpha\,dx)^{1/(p+1)}}{(\int_B |\widetilde{u}|^{p+1}|x|^\alpha\,dx)^{(q+1)/q(p+1)}} = S\Big(\int_B \widetilde{u}^{p+1}|x|^\alpha\,dx\Big)^{-1/q(p+1)} \leqslant S.
\end{aligned}$$

It follows that $\int_B \widetilde{u}^{p+1}|x|^\alpha = 1$ and $\widetilde{u}$ is a minimizer for S, so that $T(\widetilde{u}) = S|x|^\alpha \widetilde{u}^p$. Hence $\widetilde{u} = u_H$ and we conclude that u_H is a minimizer for S.

Now, up to normalization, with $-v := |x|^{-\frac{\beta}{q}}|\Delta u|^{\frac{1}{q}-1}\Delta u$ and $-w := |x|^{-\frac{\beta}{q}}|\Delta u_H|^{\frac{1}{q}-1}\Delta u_H$, by Lemma 3.5 we have that $-\Delta u = |x|^\beta v^q$, $-\Delta v = |x|^\alpha u^p$, $-\Delta u_H = |x|^\beta w^q$, $-\Delta w = |x|^\alpha u_H^p$ in B and $u, v, u_H, w = 0$ on ∂B. In particular, we infer from the equations $-\Delta v = |x|^\alpha u^p$ and $-\Delta w = |x|^\alpha u_H^p$ that $w \geqslant v_H$ in $H \cap B$, cf. Lemma 3.7. Then, since, by definition, $|u - \bar{u}| = 2u_H - u - \bar{u}$ in B, we see that

$$-\Delta\big(|u - \bar{u}|\big) = |x|^\beta\big(\big(w^q - v^q\big) + \big(w^q - \bar{v}^q\big)\big) \geqslant 0 \quad \text{in } H \cap B.$$

This implies that either $u > \bar{u}$ in $H \cap B$, $u < \bar{u}$ in $H \cap B$ or else $u = \bar{u}$ in $H \cap B$. Going back to the system, we must have that either $v > \bar{v}$ in $H \cap B$, $v < \bar{v}$ in $H \cap B$ or else $v = \bar{v}$ in $H \cap B$

respectively. Fix any point $x_0 \in B$ with $u(x_0) = \max\{u(x):\ x \in B,\ |x| = |x_0|\}$. By a standard argument (cf. e.g. [2,3]), we now deduce that u and v are foliated Schwarz symmetric with respect to $e := x_0/|x_0|$. Indeed, one just needs to use the property that a continuous function u is foliated Schwarz symmetric with respect to a unit vector $e \in \mathbb{R}^N$ if and only if $u = u_H$ for every $H \in \mathcal{H}$ such that $e \in int(H)$. □

3.3. Symmetry breaking

This subsection is devoted to the proof of Theorems 1.4, 1.5 and 1.6 stated in the Introduction.

Proof of Theorem 1.4. We prove this result by comparing the radial ground critical level c_{rad} (cf. Section 2) associated to (1.1) with the ground critical level, call it $\widetilde{c}_{\alpha,\beta}$; namely, we show that under our assumptions we have that $c_{\mathrm{rad}} > \widetilde{c}_{\alpha,\beta}$. In fact, it is proved in [8, Theorem 2 c)] that

$$\widetilde{c}_{\alpha,\beta} \leqslant C_0 \alpha^{\frac{2(p+1)(q+1)-N(pq-1)}{pq-1}} \qquad (C_0 > 0,\ \beta \leqslant \alpha,\ \alpha \geqslant \alpha_0); \tag{3.8}$$

actually, in [8] it is assumed $p > 1$, $q > 1$ and $\beta < (q+1)N$ but a close inspection of their proof shows that (3.8) remains valid as long as $\beta \leqslant \alpha$ and α is sufficiently large.

On the other hand, we have shown in (2.12) that

$$c_{\mathrm{rad}} \geqslant C_1 \alpha^{\frac{(p+2)(q+1)}{pq-1}} \big(1 + \beta^{\frac{p+1}{pq-1}}\big) \quad (C_1 > 0,\ \alpha, \beta \geqslant 0). \tag{3.9}$$

Now, since $\alpha \leqslant C\beta$, by comparing (3.8) and (3.9) for large values of α we get the desired conclusion by observing that $N(pq-1) > pq-1$ by assumption. □

Proof of Theorem 1.5. Again we denote by $c_{\alpha,\beta}$ the radial ground state level, associated to a least energy radial solution $(u_{\alpha,\beta}, v_{\alpha,\beta})$. We argue by contradiction and assume that $(u_{\alpha,\beta}, v_{\alpha,\beta})$ is a ground state solution of the problem. We claim that then there exists $C_0 > 0$ such that

$$c_{\alpha,\beta} \leqslant C_0 \bigg(\int_B |\nabla u_{\alpha,\beta}|^2\,dx\bigg)^{1/2} \bigg(\int_B \frac{v_{\alpha,\beta}^2}{|x|^2}\,dx\bigg)^{1/2}, \quad \forall \alpha, \beta \geqslant 0; \tag{3.10}$$

we stress that the constant $C_0 = C_0(p,q,N)$ in (3.10) is independent of α and β. We postpone to the end of the subsection the proof of this inequality, cf. Proposition 3.15. Notice that our method requires $p \geqslant 1$, $q \geqslant 1$ and $N \geqslant 3$.

On the other hand, we have shown in Lemma 2.17 that there exists $\alpha_0 > 0$ such that

$$\bigg(\int_B |\nabla u_{\alpha,\beta}|^2\,dx\bigg)^{1/2} \bigg(\int_B \frac{v_{\alpha,\beta}^2}{|x|^2}\,dx\bigg)^{1/2} \leqslant \frac{C_1}{1+\sqrt{\beta}} c_{\alpha,\beta}, \quad \forall \alpha \geqslant \alpha_0,\ \beta \geqslant 0 \tag{3.11}$$

for some positive constant $C_1 = C_1(p,q,N)$, as long as $\beta/\alpha \to 0$. In fact, it is sufficient to have $\beta/\alpha \leqslant \delta$ for some $\delta < 1/(p+2)$.

By comparing (3.10) and (3.11) we obtain a contradiction provided β is taken sufficiently large. □

Proof of Theorem 1.6. Case (i) is proved as in Theorem 1.4 by taking the estimates (3.8) and (3.9) into account. The proof of case (ii) is included in the above proof of Theorem 1.5. □

We end this subsection by establishing the inequality (3.10). This will be achieved in Proposition 3.15 after we prove some preliminary results. Let us first introduce some notations. In the sequel, we assume that (H2) holds and also that $N \geqslant 3$, $pq > 1$, $p \geqslant 1$ and $q \geqslant 1$. For given $\alpha, \beta \geqslant 0$, let (u_α, v_α) be any radially symmetric solution pair of the system.

As explained in Section 4, it is not restrictive to assume further that $\max\{p, q\}(N-2) < N+2$. In this case, the energy functional

$$I(u, v) = \int\limits_B \langle \nabla u, \nabla v \rangle \, dx - \int\limits_B |x|^\alpha F(u) \, dx - \int\limits_B |x|^\beta G(v) \, dx,$$

with $F(u) = \frac{1}{p+1}|u|^{p+1}$, $G(v) = \frac{1}{q+1}|v|^{q+1}$, is of class C^2 in the space $H_0^1(B) \times H_0^1(B)$. We will write $\|u\|^2 := \int_B |\nabla u|^2 \, dx$, $\langle u, v \rangle := \int_B \langle \nabla u, \nabla v \rangle \, dx$ and will denote by $c_{\alpha,\beta}$ the (up to a multiplicative factor) critical level which corresponds to (u_α, v_α), namely

$$c_{\alpha,\beta} = \int\limits_B |x|^\alpha f(u_{\alpha,\beta}) u_{\alpha,\beta} \, dx = \int\limits_B |x|^\beta g(v_{\alpha,\beta}) v_{\alpha,\beta} \, dx = \int\limits_B \langle \nabla u_{\alpha,\beta}, \nabla v_{\alpha,\beta} \rangle \, dx, \tag{3.12}$$

with $f(s)s = |s|^{p+1}$, $g(s)s = |s|^{q+1}$. In the proofs of the lemmas below, in order to simplify the notations we occasionally drop the subscripts α, β in $(u_{\alpha,\beta}, v_{\alpha,\beta})$. We observe that

$$pq > 1 \quad \Leftrightarrow \quad p + q - 2 > \frac{(p-1)^2}{p} \quad \Leftrightarrow \quad p + q - 2 > \frac{(q-1)^2}{q}. \tag{3.13}$$

Lemma 3.9. *For any $\lambda > 0$, $\mu, \mu' \in [0, 1]$, $\phi \in H_0^1(B)$, we have that*

$$\begin{aligned} &-I''(u_{\alpha,\beta}, v_{\alpha,\beta})(u_{\alpha,\beta} + \phi, v_{\alpha,\beta} - \phi/\lambda)(u_{\alpha,\beta} + \phi, v_{\alpha,\beta} - \phi/\lambda) \\ &\quad \geqslant \left(p + q - 2 - \mu^2 \frac{(p-1)^2}{p} - \mu'^2 \frac{(q-1)^2}{q} \right) c_{\alpha,\beta} \\ &\qquad - \left((p-1)^2 (1-\mu)^2 \lambda \|v_{\alpha,\beta}\|^2 + (q-1)^2 \left(1-\mu'\right)^2 \frac{\|u_{\alpha,\beta}\|^2}{\lambda} \right). \end{aligned}$$

Proof. By direct computation we see that the second derivative above is given by

$$\frac{2}{\lambda}\|\phi\|^2 - 2\langle u, v \rangle + 2\left\langle \left(\frac{u}{\lambda} - v \right), \phi \right\rangle + \int\limits_B |x|^\alpha f'(u)(u+\phi)^2 + \int\limits_B |x|^\beta g'(v)\left(v - \frac{\phi}{\lambda} \right)^2,$$

that is, multiplying the equations of the system respectively by ϕ/λ and ϕ and using (3.12),

$$\frac{2}{\lambda}\|\phi\|^2 + (p+q-2)c_{\alpha,\beta} + p\int_B |x|^\alpha \frac{f(u)}{u}\phi^2 + q\int_B |x|^\beta \frac{g(v)}{v}\frac{\phi^2}{\lambda^2}$$
$$+ 2(p-1)\int_B |x|^\alpha f(u)\phi - 2(q-1)\int_B |x|^\beta g(v)\frac{\phi}{\lambda}.$$

By writing

$$2(p-1)\int_B |x|^\alpha f(u)\phi = 2(p-1)\mu\int_B |x|^\alpha \frac{f(u)}{u}u\phi + 2(p-1)(1-\mu)\langle v,\phi\rangle,$$

we deduce that

$$\left|2(p-1)\int_B |x|^\alpha f(u)\phi\right| \leqslant p\int_B |x|^\alpha \frac{f(u)}{u}\phi^2 + \mu^2\frac{(p-1)^2}{p}c_{\alpha,\beta} + \frac{1}{\lambda}\|\phi\|^2$$
$$+ (p-1)^2(1-\mu)^2\lambda\|v\|^2.$$

Similarly, we obtain

$$\left|2(q-1)\int_B |x|^\beta g(v)\frac{\phi}{\lambda}\right| \leqslant q\int_B |x|^\beta \frac{g(v)}{v}\frac{\phi^2}{\lambda^2} + \mu'^2\frac{(q-1)^2}{q}c_{\alpha,\beta} + \frac{1}{\lambda}\|\phi\|^2$$
$$+ (q-1)^2\bigl(1-\mu'\bigr)^2\frac{\|u\|^2}{\lambda},$$

and the claim follows. □

We denote by $h = h(\sigma)$ the first non-constant spherical harmonic in dimension N. Namely,

$$-\Delta_{S^{N-1}}h = (N-1)h, \qquad \int_{S^{N-1}} h = 0 \quad \text{and} \quad \int_{S^{N-1}} h^2 = 1.$$

Lemma 3.10. *For any radially symmetric function $\phi \in H_0^1(B)$, we have that*

$$-I''(u_{\alpha,\beta}, v_{\alpha,\beta})(u_{\alpha,\beta}h + \phi h, v_{\alpha,\beta}h - \phi h/\lambda)(u_{\alpha,\beta}h + \phi h, v_{\alpha,\beta}h - \phi h/\lambda)$$
$$= -I''(u_{\alpha,\beta}, v_{\alpha,\beta})(u_{\alpha,\beta} + \phi, v_{\alpha,\beta} - \phi/\lambda)(u_{\alpha,\beta} + \phi, v_{\alpha,\beta} - \phi/\lambda)$$
$$+ 2(N-1)\left(\frac{1}{\lambda}\int_B \frac{\phi^2}{|x|^2} - \int_B \frac{u_{\alpha,\beta}v_{\alpha,\beta}}{|x|^2} + \int_B \frac{(\frac{u_{\alpha,\beta}}{\lambda} - v_{\alpha,\beta})\phi}{|x|^2}\right).$$

Proof. As all the functions involved except for h are radially symmetric, we have that

$$\int_B |x|^\alpha f'(u)h^2(u+\phi)^2 = \int_B |x|^\alpha f'(u)(u+\phi)^2,$$

$$\int_B |x|^\beta g'(v)h^2\left(v-\frac{\phi}{\lambda}\right)^2 = \int_B |x|^\beta g'(v)\left(v-\frac{\phi}{\lambda}\right)^2.$$

Using the definition of h we also deduce

$$\langle \chi h, \xi h\rangle = \langle \chi, \xi\rangle + (N-1)\int_B \frac{\chi\xi}{|x|^2},$$

for any radial functions $\chi, \xi \in H_0^1(B)$. The conclusion follows easily from these identities. □

Lemma 3.11. *There exists $\lambda = \lambda_{\alpha,\beta}$, depending on $(u_{\alpha,\beta}, v_{\alpha,\beta})$, such that, if*

$$I''(u_{\alpha,\beta}, v_{\alpha,\beta})(u_{\alpha,\beta}h+\phi h, v_{\alpha,\beta}h-\phi h/\lambda)(u_{\alpha,\beta}h+\phi h, v_{\alpha,\beta}h-\phi h/\lambda) \geqslant 0$$

for some radially symmetric function $\phi = \phi(r) \in H_0^1(B)$, then

$$c_{\alpha,\beta} \leqslant C_0\|u_{\alpha,\beta}\|\left(\int_B \frac{v_{\alpha,\beta}^2}{|x|^2}\,dx\right)^{1/2}$$

for some constant $C_0 = C_0(p,q,N) > 0$.

Proof. By combining the previous two lemmas (we take $\mu' = 0$ and $\mu = 1$ in Lemma 3.9), we deduce that

$$\begin{aligned}\left(p+q-2-\frac{(p-1)^2}{p}\right)c_{\alpha,\beta} &\leqslant (q-1)^2\frac{\|u\|^2}{\lambda}\\ &\quad + 2(N-1)\left(-\frac{1}{\lambda}\int_B \frac{\phi^2}{|x|^2} + \int_B \frac{uv}{|x|^2} - \int_B \frac{(\frac{u}{\lambda}-v)\phi}{|x|^2}\right).\end{aligned}$$

This implies that

$$C_1\left(p+q-2-\frac{(p-1)^2}{p}\right)c_{\alpha,\beta} \leqslant \lambda\int_B \frac{v^2}{|x|^2} + \frac{1}{\lambda}\left(\|u\|^2 + \int_B \frac{u^2}{|x|^2}\right),$$

for some constant $C_1 > 0$ depending on p, q and N, and so, using the Hardy inequality and recalling (3.13), we get

$$C_2 c_{\alpha,\beta} \leqslant \lambda\int_B \frac{v^2}{|x|^2} + \frac{1}{\lambda}\|u\|^2, \quad \forall\lambda > 0.$$

We minimize this expression with respect to λ, that is we take

$$\lambda = \frac{\|u\|}{\sqrt{\int_B \frac{v^2}{|x|^2}}}, \tag{3.14}$$

and we end up with

$$C_3 c_{\alpha,\beta} \leqslant 2\|u\| \left(\int\limits_B \frac{v^2}{|x|^2} \right)^{1/2}. \qquad \square$$

Remark 3.12. Similarly, by interchanging the roles of μ and μ' in the above proof, we can choose λ in such a way that $c_{\alpha,\beta} \leqslant C_0 \|v_{\alpha,\beta}\| (\int_B u_{\alpha,\beta}^2/|x|^2)^{1/2}$.

We now use the family of functionals $\bar{I}_\lambda : H_0^1(B) \to \mathbb{R}$ (cf. (4.1)) defined by

$$\bar{I}_\lambda(w) = \sup\{ I(\lambda w + \psi, w - \psi/\lambda) \colon \psi \in H_0^1(B) \},$$

where $\lambda > 0$. Its second derivative at a critical point $(u_{\alpha,\beta}, v_{\alpha,\beta})$ is computed in both [4,26]; for any $w \in H_0^1(B)$, by letting $w_{\alpha,\beta} := (u_{\alpha,\beta} + \lambda v_{\alpha,\beta})/2\lambda$, we have

$$\begin{aligned} \bar{I}''_\lambda(w_{\alpha,\beta}) ww &= I''(u_{\alpha,\beta}, v_{\alpha,\beta})(\lambda w + \phi_w, w - \phi_w/\lambda)(\lambda w, w) \\ &= I''(u_{\alpha,\beta}, v_{\alpha,\beta})(\lambda w + \phi_w, w - \phi_w/\lambda)(\lambda w + \phi_w, w - \phi_w/\lambda) \\ &= \sup\{ I''(u_{\alpha,\beta}, v_{\alpha,\beta})(\lambda w + \phi, w - \phi/\lambda)(\lambda w + \phi, w - \phi/\lambda) \colon \phi \in H_0^1(B) \}, \end{aligned}$$

where we have denoted by ϕ_w the unique solution of the linear problem

$$\begin{aligned} -2\Delta\phi &+ \left(\lambda |x|^\alpha f'(u_{\alpha,\beta}) + \frac{1}{\lambda} |x|^\beta g'(v_{\alpha,\beta}) \right) \phi \\ &= \left(|x|^\beta g'(v_{\alpha,\beta}) - \lambda^2 |x|^\alpha f'(u_{\alpha,\beta}) \right) w, \quad \phi \in H_0^1(B). \end{aligned} \tag{3.15}$$

Lemma 3.13. *There exists $\lambda = \lambda_{\alpha,\beta}$, depending on $(u_{\alpha,\beta}, v_{\alpha,\beta})$, such that, if*

$$\bar{I}''_\lambda(w_{\alpha,\beta})(w_{\alpha,\beta} h)(w_{\alpha,\beta} h) \geqslant 0$$

then

$$c_{\alpha,\beta} \leqslant C_0 \|u_{\alpha,\beta}\| \left(\int\limits_B \frac{v_{\alpha,\beta}^2}{|x|^2}\, dx \right)^{1/2}$$

for some constant $C_0 = C_0(p, q, N) > 0$.

Proof. If we let $w = w_{\alpha,\beta} h$ in (3.15) then the solution ϕ_w can be written as $\phi_w = \phi_0(r) h(\sigma)$ for some radially symmetric function $\phi_0 \in H_0^1(B)$. Indeed, in order to simplify the notations, let us write $a(x)=(\lambda|x|^{\alpha}f'(u_{\alpha,\beta})+\frac{1}{\lambda}|x|^{\beta}g'(v_{\alpha,\alpha}))$ and $b(x)=(|x|^{\beta}g'(v_{\alpha,\beta})-\lambda^2|x|^{\alpha}f'(u_{\alpha,\beta}))$. Let ϕ_0 be the unique solution of the linear problem

$$-2\Delta\phi_0+2(N-1)\frac{\phi_0}{|x|^2}+a(x)\phi_0=b(x)\phi_0,\quad \phi_0\in H_0^1(B).$$

By uniqueness, ϕ_0 is radially symmetric. By letting $\phi(x):=\phi_0(r)h(\sigma)$, we see that

$$\begin{aligned}-2\Delta\phi&=-2r^{1-N}\big(r^{N-1}\phi_0'\big)'h-2\frac{\phi_0}{r^2}\Delta_{S^{N-1}}h\\&=\left(-2\Delta\phi_0+\frac{2(N-1)}{r^2}\phi_0\right)h\\&=\big(b(x)-a(x)\big)\phi_0h=\big(b(x)-a(x)\big)\phi,\end{aligned}$$

which is precisely the equation in (3.15), and this proves our previous claim.

Now, by letting $\psi_{\alpha,\beta}=\lambda w_{\alpha,\beta}-u_{\alpha,\beta}=\lambda(v_{\alpha,\beta}-w_{\alpha,\beta})$, so that $(\lambda w_{\alpha,\beta},w_{\alpha,\beta})=(u_{\alpha,\beta}+\psi_{\alpha,\beta},v_{\alpha,\beta}-\psi_{\alpha,\beta}/\lambda)$, we infer from our assumption that

$$\begin{aligned}0&\leqslant\bar I_\lambda''(w_{\alpha,\beta})(w_{\alpha,\beta}h)(w_{\alpha,\beta}h)\\&=I''(u_{\alpha,\beta},v_{\alpha,\beta})(\lambda w_{\alpha,\beta}h+\phi_0h,w_{\alpha,\beta}h-\phi_0h/\lambda)(\lambda w_{\alpha,\beta}h+\phi_0h,w_{\alpha,\beta}h-\phi_0h/\lambda)\\&=I''(u_{\alpha,\beta},v_{\alpha,\beta})(u_{\alpha,\beta}h+\phi h,v_{\alpha,\beta}h-\phi h/\lambda)(u_{\alpha,\beta}h+\phi h,v_{\alpha,\beta}h-\phi h/\lambda)\end{aligned}$$

with $\phi:=\psi_{\alpha,\beta}+\phi_0$. We may conclude thanks to Lemma 3.11. □

Lemma 3.14. *Assume $p>1$ and $q>1$. If $(u_{\alpha,\beta},v_{\alpha,\beta})$ is a ground state solution then*

$$c_{\alpha,\beta}\leqslant C_0\|u_{\alpha,\beta}\|\left(\int_B\frac{v_{\alpha,\beta}^2}{|x|^2}\,dx\right)^{1/2}$$

for some constant $C_0=C_0(p,q,N)>0$.

Proof. Let λ be given by Lemma 3.13. We will prove that if the radial solution $(u_{\alpha,\beta},v_{\alpha,\beta})$ is a ground state solution, then

$$\bar I_\lambda''(w_{\alpha,\beta})ww\geqslant 0,$$

for every $w=w_0(r)h(\sigma)$, where w_0 is an arbitrary radially symmetric function. Taking $w_0=w_{\alpha,\beta}$, the conclusion then follows from Lemma 3.13.

Given $w\in H_0^1(B)$, let $\theta(\varepsilon):=\bar I_\lambda(t(w_{\alpha,\beta}+\varepsilon w)(w_{\alpha,\beta}+\varepsilon w))$, where $t(w_{\alpha,\beta}+\varepsilon w)$ is the unique point $t>0$ such that $(w_{\alpha,\beta}+\varepsilon w)t$ lies in the Nehari manifold N_λ associated to $\bar I_\lambda$; since $p>1$ and $q>1$, the map θ_ε is well defined, as shown in [27]. Assuming that $(u_{\alpha,\beta},v_{\alpha,\beta})$ is a ground state solution, we must have that $\theta''(0)\geqslant 0$, cf. Lemma 4.1. By computing, this reads as

$$-\big(t'(w_{\alpha,\beta})w\big)\bar I_\lambda''(w_{\alpha,\beta})w_{\alpha,\beta}w_{\alpha,\beta}\leqslant\bar I_\lambda''(w_{\alpha,\beta})ww.\tag{3.16}$$

At this point we need an explicit expression for $t'(w_{\alpha,\beta})w$. By definition, for any w we have that $\bar{I}'_\lambda(t(w)w)w = 0$. By differentiating this identity along an arbitrary direction $\zeta \in H_0^1(B)$, we see that

$$\big(t'(w)\zeta\big)\bar{I}''_\lambda\big(t(w)w\big)ww = -t(w)\bar{I}''_\lambda\big(t(w)w\big)w\zeta,$$

for every $w \in H_0^1(B)$. In particular, by letting $w = w_{\alpha,\beta}$, we deduce that

$$t'(w_{\alpha,\beta})\zeta = -\frac{\bar{I}''_\lambda(w_{\alpha,\beta})w_{\alpha,\beta}\zeta}{\bar{I}''_\lambda(w_{\alpha,\beta})w_{\alpha,\beta}w_{\alpha,\beta}}, \quad \forall\zeta \in H_0^1(B).$$

From now on we fix an arbitrary radially symmetric function $w_0(r)$ and take $\zeta(x) = w_0(r)h(\sigma)$ where $h(\sigma)$ is the first non-constant spherical harmonic in dimension N. We claim that in this case, $t'(w_{\alpha,\beta})\zeta = 0$. Indeed, arguing as above, we have that

$$\begin{aligned}
\bar{I}''_\lambda(w_{\alpha,\beta})w_{\alpha,\beta}\zeta &= I''(u_{\alpha,\beta}, v_{\alpha,\beta})(\lambda w_{\alpha,\beta} + \phi_{\alpha,\beta}, w_{\alpha,\beta} - \phi_{\alpha,\beta}/\lambda)(\lambda\zeta, \zeta) \\
&= 2\lambda \int_B \langle \nabla w_{\alpha,\beta}, \nabla\zeta\rangle - \int_B |x|^\alpha f'(u_{\alpha,\beta})(\lambda w_{\alpha,\beta} + \phi_{\alpha,\beta})\lambda\zeta \\
&\quad - \int_B |x|^\beta g'(v_{\alpha,\beta})(w_{\alpha,\beta} - \phi_{\alpha,\beta}/\lambda)\zeta \\
&= \lambda \int_B \big(|x|^\alpha f(u_{\alpha,\beta}) + |x|^\beta g(v_{\alpha,\beta})\big)\zeta - \int_B |x|^\alpha f'(u_{\alpha,\beta})(\lambda w_{\alpha,\beta} + \phi_{\alpha,\beta})\lambda\zeta \\
&\quad - \int_B |x|^\beta g'(v_{\alpha,\beta})(w_{\alpha,\beta} - \phi_{\alpha,\beta}/\lambda)\zeta,
\end{aligned} \tag{3.17}$$

where $\phi_{\alpha,\beta} \in H_0^1(B)$ solves the equation in (3.15) (of course, we are taking $w = w_{\alpha,\beta}$ in the right hand side of (3.15)). By uniqueness, $\phi_{\alpha,\beta}$ is radially symmetric. So, since h, hence ζ, has mean value zero on the sphere, $t'(w_{\alpha,\beta})w = 0$ as claimed.

Going back to (3.16), we conclude that for any $w = w_0(r)h(\sigma)$, we have $\bar{I}''_\lambda(w_{\alpha,\beta})ww \geqslant 0$, as claimed. □

Proposition 3.15. *Assume* (H2)*, $pq > 1$, $p \geqslant 1$, $q \geqslant 1$, $N \geqslant 3$. If the radially symmetric solution $(u_{\alpha,\beta}, v_{\alpha,\beta})$ is a ground state solution then*

$$c_{\alpha,\beta} \leqslant C_0\|u_{\alpha,\beta}\|\bigg(\int_B \frac{v_{\alpha,\beta}^2}{|x|^2}\,dx\bigg)^{1/2}$$

for some constant $C_0 = C_0(p,q,N) > 0$.

Proof. By taking Lemma 3.14 into account, we may already assume that say, $p > 1 = q$. The only difference with respect to the previous arguments is that the map θ introduced in the proof of Lemma 3.14 must be properly defined.

To this purpose, at first we observe that, going back to Lemma 3.9 and to the choice of $\lambda = \lambda_{\alpha,\beta}$ in Lemma 3.11, there exist $C_1, C_2 > 0$ such that

$$-\bar{I}_\lambda''(w_{\alpha,\beta})w_{\alpha,\beta}w_{\alpha,\beta} \geqslant C_1 c_{\alpha,\beta} - C_2\|u_{\alpha,\beta}\|\left(\int_B \frac{v_{\alpha,\beta}^2}{|x|^2}\right)^{1/2}.$$

Therefore, in order to prove Proposition 3.15 we may assume that

$$\bar{I}_\lambda''(w_{\alpha,\beta})w_{\alpha,\beta}w_{\alpha,\beta} < 0. \tag{3.18}$$

Now, for $\varepsilon > 0$, denote $w_\varepsilon := w_{\alpha,\beta} + \varepsilon w_{\alpha,\beta}h$, where, as before, $h(\sigma)$ is the first non-constant spherical harmonic in dimension N. Define

$$\theta(t,\varepsilon) = \bar{I}_\lambda'(tw_\varepsilon)w_\varepsilon.$$

Since $\frac{\partial\theta}{\partial t}(1,0) < 0$ as shown by (3.18), the implicit function theorem implies the existence of a local diffeomorphism $\varepsilon \mapsto T(\varepsilon)$ and $\varepsilon_0 > 0$ such that for very $\varepsilon \in [0, \varepsilon_0]$,

$$\bar{I}_\lambda'\big(T(\varepsilon)w_\varepsilon\big)w_\varepsilon = 0.$$

By differentiating with respect to ε, we get

$$T'(0)\bar{I}_\lambda''(w_{\alpha,\beta})w_{\alpha,\beta}w_{\alpha,\beta} + \bar{I}_\lambda''(w_{\alpha,\beta})(w_{\alpha,\beta}h)w_{\alpha,\beta} + \bar{I}_\lambda'(w_{\alpha,\beta})(w_{\alpha,\beta}h) = 0.$$

We now recall from the expression in (3.17) that $\bar{I}_\lambda''(w_{\alpha,\beta})(w_{\alpha,\beta}h)w_{\alpha,\beta} = 0$, which implies that $T'(0) = 0$.

At last, define the function

$$\theta(\varepsilon) := \bar{I}_\lambda\big(T(\varepsilon)w_\varepsilon\big), \quad \varepsilon \in [0, \varepsilon_0].$$

Since $(u_{\alpha,\beta}, v_{\alpha,\beta})$ is a ground state solution and $T(\varepsilon)w_\varepsilon$ is a path in the Nehari manifold N_λ starting from $w_{\alpha,\beta}$, we must have that $\theta''(0) \geqslant 0$. Since $T'(0) = 0$, this in turn implies that $\bar{I}_\lambda''(w_{\alpha,\beta})(w_{\alpha,\beta}h)(w_{\alpha,\beta}h) \geqslant 0$. Therefore, the conclusion of Lemma 3.13 holds and this finishes our proof. □

Remark 3.16. In relation to the proof of Theorems 1.5 and 1.6 (ii), suppose for a moment that $\beta \geqslant 0$ in (1.1) is fixed, $\alpha \to \infty$ and denote by c_α the corresponding radial ground state level. It follows from Lemma 2.18 that

$$\left(\int_B |\nabla u_\alpha|^2\,dx\right)^{1/2}\left(\int_B \frac{v_\alpha^2}{|x|^2}\,dx\right)^{1/2} \sim c_\alpha.$$

This suggests that the method used in the proof of the quoted theorems does not apply indeed in case β is not assumed to be sufficiently large. In this situation, the information on the second derivative of the energy functional does not yield new asymptotic information on the relevant norms of (u_α, v_α). This is in great contrast with the case of a single equation, as treated by [30].

4. Appendix on variational approaches to the problem

Let us consider system (1.1) under our basic assumptions of subcriticality (H2) and superlinearity $pq > 1$. The problem of finding solutions, in particular ground state solutions, for (1.1) can be tackled in several, essentially equivalent ways; each has its own advantages and limitations.

In the present paper our main approach consists in inverting one of the equations in (1.1). The idea goes back to P.L. Lions [22], see also [10,11,16,17,21,32], and has been used recently by the authors in [5]. The presence of the weights makes the functional setting more delicate. As explained in Sections 2 and 3, this leads us to deal with the functional

$$J(u) = \frac{q}{q+1} \int_B |\Delta u|^{\frac{q+1}{q}} |x|^{-\frac{\beta}{q}}\, dx - \frac{1}{p+1} \int_B |u|^{p+1} |x|^{\alpha}\, dx,$$

over the Sobolev space $\{u \in W^{2,\frac{q+1}{q}}(B) \cap W_0^{1,\frac{q+1}{q}}(B)\colon \int_B |\Delta u|^{\frac{q+1}{q}} |x|^{-\frac{\beta}{q}}\, dx < +\infty\}$; for a given critical point u of J, the couple (u, v) with $-v := |x|^{-\frac{\beta}{q}} |\Delta u|^{\frac{1}{q}-1} \Delta u$ is a solution for (1.1). Least energy critical points for J can be found by minimizing J on the associated Nehari manifold.

We also mention the so-called *dual method* as used e.g. in [1]. In this framework, one lets $L_\alpha^{(p+1)/p}(B) := \{u \text{ measurable: } \int_B |u|^{(p+1)/p} |x|^{-\alpha/p}\, dx < \infty\} \subset L^{(p+1)/p}(B)$ and similarly for $L_\beta^{(q+1)/q}(B)$; also, $X := L_\alpha^{(p+1)/p}(B) \times L_\beta^{(q+1)/q}(B)$, $K := (-\Delta)^{-1}$ and $(Tw, w) := w_1 K w_2 + w_2 K w_1$. One then considers the C^1-functional $\overline{J} : X \to \mathbb{R}$ given by

$$\overline{J}(w) = \overline{J}(w_1, w_2) := \frac{p}{p+1} \int_B |w_1|^{(p+1)/p} |x|^{-\alpha/p}\, dx + \frac{q}{q+1} \int_B |w_2|^{(q+1)/q} |x|^{-\beta/q}\, dx - \frac{1}{2} \int_B (Tw, w)\, dx.$$

It turns out that (w_1, w_2) is a critical point of $\overline{J}$ iff (u, v) is a solution pair of (1.1), with $u := |w_1|^{\frac{1}{p}-1} w_1 |x|^{-\alpha/p}$ and $v := |w_2|^{\frac{1}{q}-1} w_2 |x|^{-\beta/q}$. In this case, we have that $\overline{J}(w_1, w_2) = \frac{pq-1}{(p+1)(q+1)} \int_B |u|^{p+1} |x|^{\alpha}\, dx$, hence (w_1, w_2) is a least energy critical point of $\overline{J}$ iff (u, v) is a ground state solution of the system. We observe that $\overline{J}$ is no longer a strongly indefinite functional; in particular, the mountain-pass theorem and its variants can be applied to this functional (in [1] it is further assumed that $p > 1$ and $q > 1$, but the general case $pq > 1$ can be handled in a similar fashion).

On the other hand, the so-called *direct method* consists in finding critical points of the (strongly indefinite) functional

$$I(u, v) = \int_B \langle \nabla u, \nabla v \rangle\, dx - \int_B \left(\frac{1}{p+1} |u|^{p+1} |x|^{\alpha} + \frac{1}{q+1} |v|^{q+1} |x|^{\beta} \right) dx,$$

defined in a convenient space of functions. The natural Sobolev space $W_0^{1,s}(B) \times W_0^{1,\frac{s}{s-1}}(B)$ was mentioned in Section 3. Instead, one can replace the domain of I by a Hilbert space by working with fractional Sobolev spaces, as in [13], provided $\max\{p, q\}(N-4) < N+4$ (which is the case if (H2) holds and moreover $p, q > 1$); in fact, by using a canonical isomorphism between these spaces and the space $H_0^1(B)$, one can work in the Sobolev space $H_0^1(B) \times H_0^1(B)$, at the price of having to deal with a modified, non-local functional, see [4, Section 5] for details.

In order to overcome the problem of the non-definiteness of the functional I one can proceed as in [4,27,28], see also [31]. Assuming (H2) and $p > 1$, $q > 1$, one first observes that it is in general not restrictive to assume that $\max\{p, q\}(N-2) < N+2$ so as to have I defined in the space $H := H_0^1(B) \times H_0^1(B)$. Indeed, otherwise we can truncate at infinity the nonlinear terms in (1.1). Since we have an *a priori* bound on the L^∞ norms of the ground state solutions of the truncated problem and since this bound turns out to be independent of the truncation. This shows we can work in the natural space H. We refer to [27] for the technical details.

Next we introduce the C^2-functional $\bar{I} : H_0^1(\Omega) \to \mathbb{R}$ given by

$$\bar{I}(w) := \sup\{I(w+\psi, w-\psi):\ \psi \in H_0^1(\Omega)\} = I(w+\psi_w, w-\psi_w),$$

for some unique $\psi_w \in H_0^1(\Omega)$. Then $\bar{I}'(w)\varphi = I'(w+\psi_w, w-\psi_w)(\varphi, \varphi)$, $\forall \varphi \in H_0^1(\Omega)$, and w is a critical point of $\bar{I}$ iff $(w+\psi_w, w-\psi_w)$ is a critical point of I. A nice feature of this approach is that ground state solutions for (1.1) can be found by minimizing $\bar{I}$ on the associated Nehari manifold, cf. (4.2) below; Morse index techniques can also used be used in this setting as shown in [4,26].

In Section 3 of the present paper a slight but decisive variant of this approach is used. For a given $\lambda > 0$, let

$$\bar{I}_\lambda(w) := \sup\left\{I\left(\lambda w+\psi, w-\frac{\psi}{\lambda}\right):\ \psi \in H_0^1(\Omega)\right\} = I\left(\lambda w+\psi_{\lambda,w}, w-\frac{\psi_{\lambda,w}}{\lambda}\right), \tag{4.1}$$

for some unique $\psi_{\lambda,w} \in H_0^1(\Omega)$,

$$N_\lambda := \{w \in H_0^1(\Omega),\ w \neq 0:\ \bar{I}_\lambda(w)w = 0\}, \tag{4.2}$$

and

$$c_\lambda := \inf_{N_\lambda} \bar{I}_\lambda. \tag{4.3}$$

The use of the free parameter λ gives rise to a whole family of functionals $\bar{I}_\lambda$; in each particular situation we deal with, the specific choice we make for λ may play a decisive role, as illustrated in [4,26] and also in the proof of Theorem 1.5 of the present paper. On the other hand, for each fixed λ the number c_λ corresponds to the ground state level of (1.1) and its value is therefore independent of λ.

Still in connection with the proof of Theorem 1.5, we mention that these considerations also apply to the limit situation in which one of the powers p or q in (1.1) equals 1, say $q = 1$ and $p > 1$. We state this explicitly. Let c_λ be given by (4.3).

Lemma 4.1. *Assume* (H2), $pq > 1$, $p \geqslant 1$, $q \geqslant 1$ *and let* (u_0, v_0) *be a ground state solution of the system* (1.1). *Then, for any* $\lambda > 0$, *we have that* $I(u_0, v_0) = c_\lambda$.

Proof. We focus on the case $p > 1 = q$, since the fully superlinear case ($p > 1$, $q > 1$) is treated in [27]. As explained above, without loss of generality we assume that $p(N-2) < N+2$. We show that $c_\lambda > 0$. To that purpose, let $w \in N_\lambda$; this means that $u := \lambda w + \psi_w$ and $v := w - \psi_w/\lambda$ are such that $(u, v) \neq (0, 0)$ and $I'(u, v)(u + \varphi, v - \varphi/\lambda) = 0$, $\forall \varphi \in H_0^1(B)$. By choosing $\varphi = 0$ we see that

$$2 \int_B \langle \nabla u, \nabla v \rangle \, dx = \int_B |x|^\alpha |u|^{p+1} \, dx + \int_B |x|^\beta v^2 \, dx > 0. \tag{4.4}$$

Then, by letting $\varphi = -u$, we obtain

$$\begin{aligned} \frac{1}{\lambda} \int_B |\nabla u|^2 \, dx &\leqslant \int_B \left\langle \nabla u, \nabla \left(v + \frac{u}{\lambda} \right) \right\rangle dx \\ &= \int_B |x|^\beta v^2 \, dx + \frac{1}{\lambda} \int_B |x|^\beta uv \, dx \leqslant \varepsilon \int_B u^2 \, dx + \left(1 + \frac{1}{\lambda^2 \varepsilon} \right) \int_B v^2 \, dx, \end{aligned}$$

for any small $\varepsilon > 0$ and so, by using twice the Poincaré inequality,

$$\int_B |\nabla u|^2 \, dx \leqslant C \int_B v^2 \, dx \leqslant C' \int_B |\nabla v|^2 \, dx. \tag{4.5}$$

Next we choose $\varphi = \lambda v$ and get that

$$\lambda \int_B |\nabla v|^2 \, dx \leqslant \int_B \langle \nabla v, \nabla (u + \lambda v) \rangle \, dx = \lambda \int_B |x|^\alpha |u|^{p-1} uv \, dx + \int_B |x|^\alpha |u|^{p+1} \, dx$$

and so, by the Hölder and Sobolev inequalities, we infer

$$\int_B |\nabla v|^2 \, dx \leqslant C \left(\int_B |x|^\alpha |u|^{p+1} \, dx \right)^{p/(p+1)} \left(\int_B |\nabla v|^2 \, dx \right)^{1/2} + C \int_B |x|^\alpha |u|^{p+1} \, dx. \tag{4.6}$$

By combining (4.6), (4.5) and the Sobolev inequality, we deduce that $\int_B |\nabla v|^2 \, dx \geqslant \mu > 0$ with μ independent of (u, v). Going back to (4.6) we deduce that

$$\int_B |x|^\alpha |u|^{p+1} \, dx \geqslant \mu' > 0.$$

Finally, by recalling (4.4), we deduce that

$$\bar{I}_\lambda(w) = I(u, v) = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_B |x|^\alpha |u|^{p+1} \, dx \geqslant \mu' \left(\frac{1}{2} - \frac{1}{p+1} \right) > 0,$$

as intended.

Once we know that $c_\lambda > 0$ and since N_λ is a natural constraint for $\bar{I}_\lambda$ (see [27, Lemma 2.2]), a standard application of the Ekeland variational principle insures that c_λ is indeed a critical value for $\bar{I}_\lambda$. Clearly, in this case it corresponds to the least critical value of $\bar{I}_\lambda$, thus also of the original functional I. □

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